Periodic Jacobi operator with finitely supported perturbation on the half-lattice

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Abstract

We consider a periodic Jacobi operator J with finitely supported perturbations on the half-lattice. We describe all eigenvalues and resonances of J and give their properties. We solve the inverse resonance problem: we prove that the mapping from finitely supported perturbations to the Jost functions is one-to-one and onto, we show how the Jost functions can be reconstructed from all eigenvalues, resonances and from the set of zeros of $S(\lambda)-1$, where $S(\lambda)$ is the scattering matrix.

1 Introduction.

We consider a Jacobi operator $J = J^0 + V$ on the half-lattice $\mathbb{N} = \{1, 2, 3, ..\}$. Here the unperturbed operator J^0 is a periodic Jacobi operator given by

$$(J^0 y)_n = a^0_{n-1} y_{n-1} + a^0_n y_{n+1} + b^0_n y_n, \qquad n \geqslant 1, \qquad y_0 = 0, \tag{1.1}$$

where $y = (y_n)_1^\infty \in \ell^2 = \ell^2(\mathbb{N})$ and the $q-$periodic coefficients $a^0_n, b^0_n \in \mathbb{R}$ satisfy

$$a^0_n = a^0_{n+q} > 0, \quad b^0_n = b^0_{n+q}, \quad n \in \mathbb{N} = \{1, 2, 3....\}, \qquad \prod_{j=1}^{q} a^0_j = 1, \quad q \geqslant 2. \tag{1.2}$$

The perturbation operator V is the finitely supported Jacobi operator given by

$$(Vy)_n = \begin{cases} u_{n-1} y_{n-1} + u_n y_{n+1} + v_n y_n, & \text{if} \quad 1 \leqslant n \leqslant p, \quad y_0 = 0, \\ u_p y_p, & \text{if} \quad n = p+1, \\ 0, & \text{if} \quad n \geqslant p+2, \quad p \geqslant 1. \end{cases} \tag{1.3}$$

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We parameterize V by the vector $(u,v)\in\mathbb{R}^{2p}$ and let (u,v) belong to the class $\mathfrak{X}_\nu$ given by

$$\mathfrak{X}_\nu = \Big\{(u,v)\in\mathbb{R}^{2p}:\ a_n^0+u_n>0,\quad n=1,...,p,\quad u_p\neq 0\Big\}\quad \text{if}\quad \nu=2p, \tag{1.4}$$

$$\mathfrak{X}_\nu = \Big\{(u,v)\in\mathbb{R}^{2p}:\ a_n^0+u_n>0,\quad n=1,...,p,\quad v_p\neq 0,\ u_p=0\Big\}\quad \text{if}\quad \nu=2p-1. \tag{1.5}$$

We rewrite J in the form

$$(Jy)_n = a_{n-1}y_{n-1}+a_ny_{n+1}+b_ny_n,\qquad n\geqslant 1,\qquad y_0=0, \tag{1.6}$$

with the coefficients a_n, b_n given by

$$a_n=\begin{cases} a_n^0+u_n>0 & \text{if}\quad n\leqslant p,\\ a_n^0 & \text{if}\quad n\geqslant p+1,\end{cases}\qquad b_n=\begin{cases} b_n^0+v_n & \text{if}\quad n\leqslant p,\\ b_n^0 & \text{if}\quad n\geqslant p+1.\end{cases} \tag{1.7}$$

The corresponding Jacobi matrices have the forms

$$J^0=\begin{pmatrix} b_1^0 & a_1^0 & 0 & 0 & ...\\ a_1^0 & b_2^0 & a_2^0 & 0 & ...\\ 0 & a_2^0 & b_3^0 & a_3^0 & ...\\ 0 & 0 & a_3^0 & b_4^0 & ...\\ 0 & 0 & 0 & a_4^0 & ...\\ ... & ... & ... & ... & ...\end{pmatrix},\qquad J=\begin{pmatrix} b_1 & a_1 & 0 & 0 & ...\\ a_1 & b_2 & a_2 & 0 & ...\\ 0 & a_2 & b_3 & a_3 & ...\\ 0 & 0 & a_3 & b_4 & ...\\ 0 & 0 & 0 & a_4 & ...\\ ... & ... & ... & ... & ...\end{pmatrix}. \tag{1.8}$$

Note that the $n=1$ case in (1.6) can be thought of as forcing the Dirichlet condition $y_0=0$. Thus, eigenfunctions must be non-vanishing at $n=1$ and eigenvalues must be simple.

The spectrum of J^0 consists of an absolutely continuous part $\sigma_{\rm ac}(J^0)=\bigcup_1^q\sigma_j$ plus at most one eigenvalue in each non-empty gap $\gamma_j, j=1,...,q-1$, where the bands σ_j and the gaps γ_j are given by

$$\sigma_j=[\lambda_{j-1}^+,\lambda_j^-],\quad j=1,\dots,q,\quad \gamma_j=(\lambda_j^-,\lambda_j^+),\qquad j=1,\dots,q-1,$$
$$\lambda_0^+<\lambda_1^-\leqslant\lambda_1^+<.....<\lambda_{q-1}^-\leqslant\lambda_{q-1}^+<\lambda_q^-. \tag{1.9}$$

We introduce the infinite gaps

$$\gamma_0=(-\infty,\lambda_0^+),\qquad \gamma_q=(\lambda_q^+,+\infty).$$

Let $\varphi=(\varphi_n(\lambda))_1^\infty$ and $\vartheta=(\vartheta_n(\lambda))_1^\infty$ be fundamental solutions for the equation

$$a_{n-1}^0y_{n-1}+a_n^0y_{n+1}+b_n^0y_n=\lambda y_n,\qquad \lambda\in\mathbb{C}, \tag{1.10}$$

satisfying the conditions $\vartheta_0=\varphi_1=1$ and $\vartheta_1=\varphi_0=0$. Here and below $a_0^0=a_q^0$. Introduce the Lyapunov function Δ by

$$\Delta=\frac{\varphi_{q+1}+\vartheta_q}{2}. \tag{1.11}$$

It is known that $\Delta(\lambda)$ is a polynomial of degree q and $\lambda_j^{\pm}, j = 1, ..., q$, are the zeros of the polynomial $\Delta^2(\lambda) - 1$ of degree $2q$. Note that $\Delta(\lambda_j^{\pm}) = (-1)^{q-j}$. In each "gap" $[\lambda_j^-, \lambda_j^+]$ there is one simple zero of polynomials $\varphi_q, \dot{\Delta}, \vartheta_{q+1}$. Here and below $\dot{f}$ denotes the derivative of $f = f(\lambda)$ with respect to $\lambda : \dot{f} \equiv \partial_\lambda f \equiv f'(\lambda)$.

Let Γ denote the complex plane cut along the segments σ_j (1.9): $\Gamma = \mathbb{C} \setminus \sigma_{\rm ac}(J^0)$. Now we introduce the two-sheeted Riemann surface Λ of $\sqrt{1-\Delta^2(\lambda)}$ by joining the upper and lower rims of two copies of the cut plane Γ in the usual (crosswise) way. We identify the first (physical) sheet Λ_1 with Γ and the second sheet we denote by Λ_2.

Let $\widetilde{}$ denote the natural projection from Λ into the complex plane $\mathbb{C}$:

$$\lambda \in \Lambda, \quad \lambda \to \widetilde{\lambda} \in \mathbb{C}. \tag{1.12}$$

By identification of $\Gamma = \mathbb{C} \setminus \sigma_{\rm ac}(J^0)$ with Λ_1, the map $\widetilde{}$ can be also considered to be projection from Λ into the physical sheet Λ_1.

The $j-$th gap on the first physical sheet Λ_1 we will denote by γ_j^+ and the same gap but on the second nonphysical sheet Λ_2 we will denote by γ_j^- and let $\gamma_j^{\rm c}$ be the union of $\overline{\gamma_j^+}$ and $\overline{\gamma_j^-}$:

$$\gamma_j^{\rm c} = \overline{\gamma_j^+} \cup \overline{\gamma_j^-}. \tag{1.13}$$

Define the function $\Omega(\lambda) = \sqrt{1-\Delta^2(\lambda)}, \lambda \in \Lambda$, by

$$\Omega(\lambda) < 0 \quad \text{for} \quad \lambda \in (\lambda_{q-1}^+, \lambda_q^-) \subset \Lambda_1. \tag{1.14}$$

Introduce the Bloch functions $\psi_n^{\pm}$ and the Titchmarch-Weyl functions $m_{\pm}$ on Λ by

$$\psi_n^{\pm}(\lambda) = \vartheta_n(\lambda) + m_{\pm}(\lambda)\varphi_n(\lambda), \tag{1.15}$$

$$m_{\pm}(\lambda) = \frac{\phi(\lambda) \pm i\Omega(\lambda)}{\varphi_q}, \quad \phi = \frac{\varphi_{q+1} - \vartheta_q}{2},\ \lambda \in \Lambda_1. \tag{1.16}$$

The projection of all singularities of $m_{\pm}$ to the complex plane coincides with the set of zeros $\{\mu_j\}_{j=1}^{q-1}$ of polynomial φ_q. Recall that $\vartheta_n, \varphi_n, \phi$ are polynomials. Recall that any polynomial $P(\lambda)$ gives rise to a function $P(\lambda) = P(\widetilde{\lambda})$ on the Riemann surface Λ of $\sqrt{1-\Delta^2(\lambda)}$.

The perturbation V satisfying (1.3) does not change the absolutely continuous spectrum:

$$\sigma_{\rm ac}(J) = \sigma_{\rm ac}(J^0) = \bigcup_{j=1}^{q} [\lambda_{j-1}^+, \lambda_j^-]. \tag{1.17}$$

The spectrum of J consists of an absolutely continuous part $\sigma_{\rm ac}(J) = \sigma_{\rm ac}(J^0)$ plus a finite number of simple eigenvalues in each non-empty gap $\gamma_j, j = 0, ..., q$.

In the present paper we consider the properties of the eigenvalues, virtual states and resonances of the operators J^0 and J, and solve the inverse problem in terms of the resonances of J. Let $R(\lambda) = (J - \lambda)^{-1}$ denote the resolvent of J and let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\ell^2 = \ell^2(\mathbb{N})$. Then for any $f, g \in \ell^2$ the function $\langle R(\lambda)f, g\rangle$ is defined on Λ_1 outside the

poles at the bound states on the gaps γ_j^+, $j = 0, \ldots, q$. We denote the set of bound states of J by $\sigma_{\rm bs}(J)$.

Recall that we consider Dirichlet boundary condition $y_0 = 0$ in (1.6). Thus, any possible non-zero solution of $Jy = \lambda y$ must have $y_1 \neq 0$, which implies that each eigenvalue of J is simple (or else a linear combination would vanish at $n = 1$ and thus for all $n \in \mathbb{N}$).

Moreover, if $f, g \in \ell^2_{\rm comp}$, where $\ell^2_{\rm comp}$ denotes the ℓ^2 functions on $\mathbb{N}$ with a finite support, then the function $\langle R(\lambda)f, g\rangle$ has an analytic extension from Λ_1 into the Riemann surface Λ.

Definition 1. *1) A number $\lambda_0 \in \Lambda_2$ is a resonance if the function $\langle R(\lambda)f, g\rangle$ has a pole at λ_0 for some $f, g \in \ell^2_{\rm comp}$. The set of resonances is denoted $\sigma_{\rm r}(J)$. The multiplicity of the resonance is the multiplicity of the pole. If $\operatorname{Re}\lambda_0 = 0$, we call λ_0 un antibound state.*
2) A real number λ_0 such that $\Delta^2(\lambda_0) = 1$ is a virtual state if $\langle R(\lambda)f, g\rangle$ has a singularity at λ_0 for some $f, g \in \ell^2_{\rm comp}$. The set of virtual states is denoted $\sigma_{\rm vs}(J)$.
3) The state $\lambda_0 \in \Lambda$ is a bound state or a resonance or a virtual state of J.

We denote the set of all states of J by $\sigma_{\rm st}(J) = \sigma_{\rm bs}(J) \cup \sigma_{\rm r}(J) \cup \sigma_{\rm vs}(J) \subset \Lambda$.

The unperturbed Jacobi operator J_0 has one simple state λ_j in each $\gamma_j^{\rm c} = \overline{\gamma_j^+} \cup \overline{\gamma_j^-}$, $j = 1, \ldots, q-1$ (see Proposition 2.1). Here the projection of λ_j to $\mathbb{C}$ coincides with $\widetilde{\lambda}_j = \mu_j$, the zero of φ_q.

Introduce the Jost solutions $f^\pm = (f_n^\pm)_0^\infty$ and the fundamental solutions $\vartheta^+ = (\vartheta_n^+)_0^\infty, \varphi^+ = (\varphi_n^+)_0^\infty$ to the equation

$$a_{n-1}y_{n-1} + a_n y_{n+1} + b_n y_n = \lambda y_n, \qquad n \geqslant 1,$$

under the conditions

$$f_n^\pm = \psi_n^\pm, \qquad \vartheta_n^+ = \vartheta_n, \qquad \varphi_n^+ = \varphi_n, \qquad n \geqslant p+1. \tag{1.18}$$

Here and below $a_0 = a_0^0 = a_q^0$. All functions $\vartheta_n^+, \varphi_n^+, n \geqslant 0$ are polynomials. We rewrite the Jost solutions $f_n^\pm$ in the form

$$f_n^\pm = \vartheta_n^+ + m_\pm \varphi_n^+, \qquad n \geqslant 0. \tag{1.19}$$

Note that for $\lambda \in \Lambda_1$ we have $f^+(\lambda) \in \ell^2$, and $f^-(\lambda) = \overline{f^+(\overline{\lambda})}$. The functions $f_n^\pm$ and the Titchmarch-Weyl functions $m_\pm$ are meromorphic functions on Λ. Recall that the S-matrix for J, J^0 is given by

$$S(\lambda) = \frac{\overline{f_0^+(\lambda)}}{f_0^+(\lambda)} = \frac{f_0^-(\lambda)}{f_0^+(\lambda)} \quad \text{for} \quad \lambda \in \sigma_{\rm ac}(J^0). \tag{1.20}$$

We pass to the formulation of main results of the paper. Recall that if $\lambda \in \sigma_{\rm st}(J^0)$ then $\widetilde{\lambda} = \mu_j \in [\lambda_j^-, \lambda_j^+]$ for some $j = 1, ..., q-1$, where μ_j denotes the Dirichlet eigenvalue and $\varphi_q(\mu_j) = 0$. Here the projection $\widetilde{\ }$ was introduced in (1.12). We describe all states of J.

Theorem 1.1. *i) The set of all state of J has the decomposition*

$$\sigma_{st}(J) = \sigma^0(J) \cup \sigma^1(J), \tag{1.21}$$

where

$$\sigma^0(J)=\{\lambda\in\sigma_{st}(J^0):\varphi_0^+(\widetilde{\lambda})=0\},\qquad \sigma^1(J)=\{\lambda\in\Lambda:\lambda\notin\sigma_{st}(J^0),f_0^+(\lambda)=0\}.$$

Moreover, each $\lambda_0\in\sigma^0(J)$ is a simple state of J and $0<|f_0^+(\lambda_0)|<\infty$.
ii) If $\lambda_1\in\Lambda_1$ is a bound state of J, then $\lambda_2\notin\sigma_{\rm st}(J)$, where $\lambda_2\in\Lambda_2$ is the same number as λ_1 but on the second sheet.
iii) Let $\lambda_0\in\Lambda$ be a zero of f_0^+. Then $\varphi_0^+(\widetilde{\lambda}_0)\neq 0$

Remark. 1) The proof of Theorem 1.1 is given in Section 2.2.
2) A state $\lambda_0\in\sigma^0(J)$ (bound, antibound or virtual state) is not a zero of the Jost function f_0^+. Moreover, λ_0 is a simple state of both J and J^0. Such a state is a singularity of the resolvent, but it is not a singularity of the S–matrix (1.20).

In accordance with the continuous case [KS] we define the important function

$$F(\lambda)=\varphi_q(\lambda)f_0^+(\lambda)f_0^-(\lambda),\qquad \lambda\in\Lambda_1. \tag{1.22}$$

For the perturbation V with $(u,v)\in\mathfrak{X}_\nu$ we define the constants

$$c_3=c_1c_2,\qquad c_1=\frac{1}{\prod_0^p a_j},\qquad c_2=\begin{cases}c_1u_p(a_p^0+a_p) & \text{if}\quad \nu=2p,\\ c_1(a_p^0)^2v_p & \text{if}\quad \nu=2p-1.\end{cases} \tag{1.23}$$

The distribution of the states is summarized in the following theorem.

Theorem 1.2. *Let the Jacobi operator $J=J^0+V$ satisfy (1.1)–(1.3). Suppose $(u,v)\in\mathfrak{X}_\nu$, where $\nu\in\{2p,2p-1\}$. Then the following facts hold true.*
1) The function $F(\lambda),\lambda\in\Lambda_1$, is a real polynomial. Each zero of F is the projection of a state of J on the first sheet. There are no other zeros. Moreover, F satisfies

$$F(\lambda)=-a_0^0\lambda^\kappa(c_3+\mathcal{O}(\lambda^{-1})),\qquad \kappa=\nu+q-1,\qquad \lambda\to\infty, \tag{1.24}$$

here κ is a total number of states (counted with multiplicities).
2) The total number of bound and virtual states is $\geqslant 2$.
3) In each finite open "gap" $\gamma_j^c=\overline{\gamma_j^+}\cup\overline{\gamma_j^-}$, $j=1,\dots,q-1$, there is always an odd number $\geqslant 1$ of states (counted with multiplicities).
4) Let $\lambda_1<\lambda_2$ be any two bound states of J, such that $\lambda_1,\lambda_2\in\gamma_j^+$, for some $j=0,\dots,q$. Assume that there are no other eigenvalues on the interval $\Omega^+=(\lambda_1,\lambda_2)\subset\gamma_j^+$. Then there exists an odd number $\geqslant 1$ of antibound states on Ω^-, where $\Omega^-\subset\gamma_j^-\subset\Lambda_2$ is the same interval but on the second sheet, each antibound state being counted according to its multiplicity.
5) $(-1)^{q-j}\dot F(\lambda)<0$ for any $\lambda\in\gamma_j^+\cap\sigma_{\rm bs}(J)$, $j=0,1,\dots,q$.
6) If $\lambda\in\sigma_{\rm bs}(J)\cup\sigma_{\rm vs}(J)\cup\sigma^0(J)$, then λ is a simple state of J.

The proof of Theorem 1.2 follows from Lemmata 2.3–2.7.

Remark. 1) The pre-image of a zero of F is an eigenvalue or a virtual state or a resonance of J. Thus we reformulate the problem for the resolvent on the Riemann surface Λ as the problem for the polynomial F on the plane.

2) There is an even number of non-real resonances since the resonances are zeros of the real polynomial F.

3) Due to this Theorem for the operator J we define the vector-state $\mathfrak{r} = (\mathfrak{r}_n)_1^\kappa$ by

$$\begin{aligned}&\{\mathfrak{r}_j\}_{j=1}^\kappa = \sigma_{\rm st}(J), \qquad \mathfrak{r}_j \in \cup_0^q \gamma_n^+ \in \Lambda_1, \qquad \mathfrak{r}_1 < \mathfrak{r}_2 < ... < \mathfrak{r}_N, \quad N \geqslant 0,\\ &\mathfrak{r}_j \in \Lambda_2, \qquad 0 \leqslant |\mathfrak{r}_{N+1}| \leqslant |\mathfrak{r}_{N+2}| \leqslant ... \leqslant |\mathfrak{r}_\kappa|,\end{aligned} \tag{1.25}$$

and the components of $\widetilde{\mathfrak{r}}$ are repeated according to the multiplicities of $\widetilde{\mathfrak{r}}_j$ as a zero of the polynomial (1.22). Here N is the number of bound states of J.

Now we pass to the inverse resonance problem. We use the parametrization $(u, v) = (u_n, v_n)_1^p \in \mathbb{R}^{2p}$ for the perturbation V of the periodic coefficients of J^0. We suppose that all gaps are open: $\lambda_j^- < \lambda_j^+$, $j = 1, \dots, q-1$. We define the class of all Jost functions on the Riemann surface Λ as follows.

Definition 2. *For $\nu \in \mathbb{N}$, let $\mathfrak{J}_\nu$ denote the class of rational functions f on Λ of the form*

$$f = P_1 + m_+ P_2,$$

$$f(\lambda) = \begin{cases} c_1 A_p + \mathcal{O}(\lambda^{-1}) & \text{if} \quad \lambda \in \Lambda_1 \\ -\frac{c_2}{A_p}\lambda^\nu + \mathcal{O}(\lambda^{\nu-1}) & \text{if} \quad \lambda \in \Lambda_2 \end{cases} \quad \text{as} \quad \lambda \to \infty,$$

where $c_1 > 0$, $c_2 \neq 0$ and P_1 and P_2 are real polynomials (with real coefficients) of the orders $\nu - 2$ and $\nu - 1$ respectively. Here

$$A_p = \prod_{j=0}^p a_j^0. \tag{1.26}$$

Let $\sigma(f)$ be the set of all zeros of f on Λ and denote $\sigma_{\rm st}(f) = \sigma(f) \cup \sigma^0(f) \subset \Lambda$, where

$$\sigma^0(f) = \{\lambda \in \sigma_{st}(J^0) : \ P_2(\widetilde{\lambda}) = 0\}.$$

We suppose that each zero of $f(\cdot)$ on the first sheet Λ_1 is real and belongs to $\cup_0^q \overline{\gamma}_j^+$. Let

$$\sigma_{\rm bs}(f) = \sigma_{\rm st}(f) \cap \cup_0^q \gamma_j^+.$$

Define the polynomial $P(\lambda) = \varphi_q(\lambda) f(\lambda) f_-(\lambda)$, $\lambda \in \Lambda_1$, where $f_- = P_1 + m_- P_2$.
We suppose that the following properties hold true:
i) if $\lambda \in \sigma(f)$, then $P_2(\lambda) \neq 0$, i.e., $\sigma(f) \cap \sigma^0(f) = \emptyset$,
ii) $(-1)^{q-j} \dot{P}(\widetilde{\lambda}) < 0$ for any $\lambda \in \gamma_j^+ \cap \sigma_{\rm bs}(f)$, $j = 0, 1, \dots, q$,
iii) if $\lambda \in \sigma_{\rm bs}(f) \cup \sigma_{\rm vs}(f) \cup \sigma^0(f)$, where $\sigma_{\rm vs}(f) = \sigma(f) \cap (\cup_0^q \lambda_j^\pm)$, then $\widetilde{\lambda}$ is a simple zero of P.

Let $(u,v) \in \mathfrak{X}_\nu$. Then from Theorems 1.1, 1.2 and asymptotics in Section 4 it follows that the Jost function $f_0^+ \in \mathfrak{J}_\nu$ with $P_1 = \vartheta_0^+$, $P_2 = \varphi_0^+$ and $\sigma_{\rm st}(f_0^+) = \sigma_{\rm st}(J)$, $\sigma^0(f_0^+) = \sigma^0(J)$.

Now we construct the mapping $\mathscr{F} : \mathfrak{X}_\nu \to \mathfrak{J}_\nu$, $\nu \in \{2p-1, 2p\}$, by the rule:

$$(u,v) \to f_0^+, \tag{1.27}$$

i.e. to each $(u,v) \in \mathfrak{X}_\nu$ we associate $f_0^+ \in \mathfrak{J}_\nu$.

Our main inverse result is formulated in the following theorem.

Theorem 1.3. *The mapping $\mathscr{F} : \mathfrak{X}_\nu \to \mathfrak{J}_\nu$ is one-to-one and onto. Moreover, the reconstruction algorithm is specified.*

In Theorem 1.3 we solve the inverse problem for mapping $\mathscr{F}$. The solution is divided into the following three parts.

1. Uniqueness. Does the Jost function $f_0^+ \in \mathfrak{J}_\nu$ determine uniquely $(u,v) \in \mathfrak{X}_\nu$?
2. Reconstruction. Give an algorithm for recovering (u,v) from $f_0^+ \in \mathfrak{J}_\kappa$ only.
3. Characterization. Give necessary and sufficient conditions for f_0^+ to be the Jost functions for some $(u,v) \in \mathfrak{X}_\nu$.

From Theorem 1.3 it follows that any $f \in \mathfrak{J}_\nu$ is the Jost function f_0^+ for unique J with $(u,v) \in \mathfrak{X}_\nu$, and $P_1 = \vartheta_0^+$, $P_2 = \varphi_0^+$, with the asymptotics

$$\vartheta_0^+ = \frac{2a_0^2 c_2}{A_p} \lambda^{\nu-2} + \mathcal{O}\left(\lambda^{\nu-3}\right), \qquad \varphi_0^+ = -\frac{2a_0 c_2}{A_p} \lambda^{\nu-1} + \mathcal{O}\left(\lambda^{\nu-2}\right), \tag{1.28}$$

where $c_2 \neq 0$, A_p is given in (1.26) and $a_0 = a_0^0 = a_q^0$.

Now we pass to the problem of reconstruction of the Jost function f_0^+ from $\sigma_{\rm st}(J)$. Recall that $\sigma_{\rm st}(J)$ consists of the zeros of f_0^+ on Λ and the set $\sigma^0(J)$ (see Remark 2) after Theorem 1.1).

By Theorem 1.2, 1), the zeros of the polynomial F defined in (1.22) are given by $\{\widetilde{\mathfrak{r}}_j\}_{j=1}^{\kappa}$, where the set $\{\mathfrak{r}_j\}_{j=1}^{\kappa} = \sigma_{\rm st}(J)$ satisfies (1.25). The polynomial F can be uniquely reconstructed from the projection of all states $\{\widetilde{\mathfrak{r}}_j\}_{j=1}^{\kappa}$ and the constant c_3 in (1.24).

We have the following result.

Theorem 1.4. *Suppose that $(u,v) \in \mathfrak{X}_\nu$ and the polynomial F has only simple zeros. Then the Jost function f_0^+ is uniquely determined by the polynomials F and φ_0^+.*

Now the polynomial φ_0^+ can be reconstructed from its zeros and the constant c_2 in (1.28). Note that simple examples show that zeros of the polynomial φ_0^+ can be real and non-real.

We have the identity

$$\varphi_0^+ = \frac{\varphi_q}{2i\Omega}\left(f_0^+ - f_0^-\right) = \frac{\varphi_q}{2i\Omega} f_0^+ \left(1 - S\right). \tag{1.29}$$

Thus the zeros of φ_0^+ (under the conditions $\varphi_q \neq 0$ and $\Omega \neq 0$) coincide with the zeros of the function $1 - S(\lambda)$ on Λ_1 (see Lemma 2.8) and their multiplicities agree.

More precisely, let $Zeros\,(S-1) \in \Lambda_1$ denote the set of all zeros of $S(\lambda)-1$ on Λ_1 (counting the multiplicities). Let $\mu_j \in \overline{\gamma_j^+} \subset \Lambda_1$, $\varphi_q(\mu_j) = 0$, $j = 1, ..., q-1$, denote the Dirichlet eigenvalue of J_0.

From Lemma 2.8 it follows that, if

$$\big[Zeros\,(S-1) \setminus \big(\{\mu_j\}_{j=1}^{q-1} \cap \{\lambda_k^\pm\}_{k=0}^{q}\big)\big] \bigcap \big(\{\mu_j\}_{j=1}^{q-1} \cup \{\lambda_k^\pm\}_{k=0}^{q}\big) = \emptyset, \qquad (1.30)$$

then the set $Zeros\,(S-1)$ is the set of all zeros of φ_0^+. We have the following result

Theorem 1.5. *Suppose that the set of zeros $Zeros\,(S-1)$ on the first sheet Λ_1, satisfy (1.30), and each zero of polynomial F is simple. Then the Jost function f_0^+ is uniquely determined by the polynomial F, the set $Zeros\,(S-1)$ and the constant c_2.*

Theorems 1.4 and 1.5 are proved in Section 3.3.

Historical remarks. A lot of papers is devoted to the resonances for the Schrödinger operator $-\frac{d^2}{dx^2} + q(x)$ on the line $\mathbb{R}$ and the half-line $\mathbb{R}_+$ with compactly supported perturbation, see [Fr], [K4], [K5], [S],[Z], [Z1], and the references given there. Zworski [Z] obtained the first results about the distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. One of the present authors obtained the uniqueness, the recovery and the characterization of the S-matrix for the Schrödinger operator with a compactly supported potential on the real line [K4] and on the half-line [K5], see also [Z1], [BKW] concerning the uniqueness.

The problem of resonances for the Schrödinger with periodic plus compactly supported potential $-\frac{d^2}{dx^2} + p(x) + q(x)$ is much less studied: [F1], [KM], [K1], [KS]. The following results were obtained in [K1], [KS]: 1) the distribution of resonances in the disk with large radius is determined, 2) some inverse resonance problem, 3) the existence of a logarithmic resonance-free region near the real axis. Note that in our paper we use the methods from [KS], modified for the Jacobi operator J.

Finite-difference Schrödinger and Jacobi operators express many similar features. Spectral and scattering properties of infinite Jacobi matrices are much studied (see [Mo], [DS1], [DS2] and references given there). The inverse problem for periodic Jacobi operators J^0 was solved in [BGGK], [K3], [KKu], [Mo], [P] and see references therein.

The inverse resonances problem was recently solved in the case of constant background [K2]. The inverse scattering problem for asymptotically periodic coefficients was solved by Khanmamedov: [Kh1] (on the line, note that the russian versions were dated much earlier), [Kh2] (on the half-line) and Egorova, Michor, Teschl [EMT] (on the line in case of quasi-periodic background).

In our paper we apply some results from [Kh1], [Kh2] and [EMT]. There were some mistakes in the paper [EMT], [BE]. Some of them we correct in Section 2.1. However, in our context of finite rank perturbations their results still hold in the original form.

We plan to apply the results of our paper to the Schrödinger operator on nanotubes (see [IK1] and references therein). The similar methods are applied in [IK2] and [IK3] to the direct and the inverse resonance problems on the line.

Plan of the paper. In Part 2 we consider the direct problems for the Jacobi operators on the half-line. In Section 2.1 we recall some well known facts about the periodic Jacobi operators and describe the states for the periodic Jacobi operators on the half-line. We present also the revised construction of the quasi-momentum map. In Section 2.2 we consider the properties of the Jost functions and prove Theorems 1.1 and 1.2.

Part 3 is devoted to the inverse resonance problem. In Section 3.2 we recall the results of Khanmamedov on the inverse scattering problem on the half-line which we apply in Section 3.3 and prove the inverse results.

In Part 4 we collect the asymptotics of the Jost functions which we need in the proofs.

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2 Direct problem

2.1 Unperturbed Jacobi operators J^0.

We need some known properties of the $q-$periodic Jacobi operator J^0 on $\mathbb{N}$ (see [P], [T], [Kh1]). Recall that the fundamental solutions $\varphi = (\varphi_n)_0^\infty$ and $\vartheta = (\vartheta_n)_0^\infty$ and the Lyapunov function Δ were defined in the Introduction. The spectrum of J^0 consists of an absolutely continuous part $\sigma_{\rm ac}(J^0) = \bigcup_1^q \sigma_j$ plus at most one eigenvalue in each non-empty gap γ_j, $j = 1, ..., q-1$, where the bands σ_j and the gaps γ_j are given by (1.9).

If there are exactly $N \geqslant 1$ nondegenerate gaps in the spectrum of $\sigma_{\rm ac}(J^0)$, then the operator J^0 has exactly N states; the closed gaps $\gamma_n = \emptyset$ do not contribute to any states. In particular, if all $\gamma_j = \emptyset, j \geqslant 1$, then $q = 1$ (see [BGGK], [KKu], [K3]) and J^0 has no states. A more detailed description of the states of J^0 is given in Proposition 2.1 below.

In each finite "gap" $[\lambda_j^-, \lambda_j^+]$, $j = 1, \ldots, q-1$, there is one simple zero of polynomials $\varphi_q(\lambda)$, $\dot\Delta(\lambda)$, $\vartheta_{q+1}(\lambda)$. Here $\lambda_1^\pm, \ldots, \lambda_{q-1}^\pm$ are all endpoints of the bands, see (1.9). Note that $\Delta(\lambda_j^\pm) = (-1)^{q-j}$. The sequence of zeros of the polynomial $\Delta^2 - 1$ of degree $2q$ can be enumerated by $(\lambda_j^\pm)_0^q$, $\lambda_0^+ = \lambda_0^-$, $\lambda_q^+ = \lambda_q^-$. We have

$$\varphi_q = a_0^0 \prod_{j=1}^{q-1} (\lambda - \mu_j), \qquad \vartheta_{q+1} = -a_0^0 \prod_{j=1}^{q-1} (\lambda - \nu_j),$$

$$\Delta^2 - 1 = \frac{1}{4}(\lambda - \lambda_0^+)(\lambda - \lambda_q^-) \prod_{j=1}^{q-1} (\lambda - \lambda_j^-)(\lambda - \lambda_j^+),$$

where $\mu_j \in [\lambda_j^-, \lambda_j^+]$ are the zeros of φ_q and $\nu_j \in [\lambda_j^-, \lambda_j^+]$ are the zeros of ϑ_{q+1} (Dirichlet or Neumann eigenvalues). We put

$$A = A_q = \prod_{j=1}^{q} a_j^0 = 1, \qquad B = \sum_{j=1}^{q} b_j^0.$$

Note the following asymptotics:

$$\varphi_q = a_0^0 \lambda^{q-1} + \mathcal{O}(\lambda^{q-2}), \qquad \Delta(\lambda) = \frac{z^q + z^{-q}}{2} = \frac{\lambda^q}{2} + \mathcal{O}(\lambda^{q-1}) \text{ as } \lambda \to \infty. \tag{2.1}$$

Here the function $z = z(\lambda) = e^{i\varkappa(\lambda)}$ is explained later in this section and $\varkappa(\lambda)$ is the quasi-momentum satisfying (2.7).

Recall that Γ is the complex λ-plane with cuts along the segments σ_j, $j = 1, 2, \ldots, q$. Γ will be identified with the first sheet Λ_1. We use the standard definition of the root: $\sqrt{1} = 1$ and fix the branch of the function $\sqrt{\Delta^2(\lambda) - 1}$ on Λ by demanding $\sqrt{\Delta^2(\lambda) - 1} < 0$ for $\lambda > \lambda_q^-, \lambda \in \Gamma$ (in accordance with (1.14)). We define the first ξ_+ and the second ξ_- Floquet multipliers on the plane Λ_1 or Γ by

$$\xi_\pm(\lambda) = \Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 1}, \qquad \lambda \in \Lambda_1.$$

By our choice of the branch we have $|\xi_+(\lambda)| < 1$, $|\xi_-(\lambda)| > 1$ and

$$\sqrt{\Delta^2(\lambda) - 1} = -\frac{1}{2}\sqrt{\lambda - \lambda_0^+}\sqrt{\lambda - \lambda_q^-}\prod_{j=1}^{q-1}\sqrt{\lambda - \lambda_j^-}\sqrt{\lambda - \lambda_j^+}. \tag{2.2}$$

for all $\lambda \in \Lambda_1$. The functions $\xi_\pm(\lambda)$ are continuous up to the boundary $\partial\Lambda_1$ and $|\xi_\pm(\lambda)| = 1$ for $\lambda \in \partial\Lambda_1$. Moreover for $\lambda \in \Lambda_1$,

$$\xi^\pm(\lambda) = (2\Delta(\lambda))^{\mp 1}\left(1 + \mathcal{O}\left(\lambda^{-2q}\right)\right) = \lambda^{\mp q}\left(1 \pm \frac{B}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right).$$

For two sequences $x = (x_n)_1^\infty, y = (y_n)_1^\infty$ we introduce the unperturbed Wronskian by

$$\{x, y\}_n^0 = a_n^0(x_n y_{n+1} - x_{n+1} y_n). \tag{2.3}$$

Using that $\{x, y\}_n^0$ is independent of n for two solutions of (1.10) and putting $x = \vartheta$, $y = \varphi$, we apply the conditions $\vartheta_0 = \varphi_1 = 1$, $\vartheta_1 = \varphi_0 = 0$ and obtain

$$1 - \Delta^2 + \phi^2 = 1 - \varphi_{q+1}\vartheta_q = -\varphi_q \vartheta_{q+1}. \tag{2.4}$$

Thus, we get

$$m_+ m_- = -\frac{\vartheta_{q+1}}{\varphi_q}. \tag{2.5}$$

This identity considered at zeros of polynomial $\varphi_q(\lambda)$ of degree $q - 1$ shows: if one of the solutions $\psi_n^\pm(\lambda)$ is regular, then the other has simple poles, one in each finite gap γ_n, $n = 1, \ldots, q - 1$.

Equation (1.1) has two Bloch solutions $\psi_n^\pm = \psi_n^\pm(\lambda)$ which satisfy $\psi_{kq}^\pm = \xi_\pm^k$, $k \in \mathbb{Z}$, and at the end points of the gaps we have $|\psi_{kq}^\pm(\lambda_n^\pm)| = 1$. As for any $\lambda \in \Lambda_1$ we have $\psi^+ \in \ell^2(\mathbb{N})$, then functions $\psi^\pm(\lambda)$ are the Floquet solutions for (1.1).

Now we consider the spectrum of the half-infinite Jacobi matrix J^0 defined by (1.8) or (1.6) with coefficients a_j^0, b_j^0, $j \in \mathbb{N}$, verifying (1.6).

Proposition 2.1 (States of J^0). *The unperturbed operator J^0 has absolutely continuous spectrum (1.17): $\sigma_{\rm ac}(J^0) = \cup_{j=1}^q \sigma_j$ and one simple state λ_j in each $\gamma_j^{\rm c} = \overline{\gamma_j^+} \cup \overline{\gamma_j^-}$, $j = 1, \dots, q-1$. Here the projection of λ_j on $\mathbb{C}$ coincides with $\widetilde{\lambda}_j = \mu_j$, the zero of φ_q.*

Proof. The kernel of the resolvent of J^0 is given by

$$R^0(n,m) = -\frac{\varphi_n \psi_m^+}{\{\varphi, \psi^+\}} = \frac{\varphi_n \psi_m^+}{a_0^0}, \ n < m,$$

since $\{\varphi, \psi^+\} = -a_0^0$. According to Lemma 2.2 (see Section 2.2), the bound states (resonances) are the poles of $\mathscr{R}_n^0 = \psi_n^+(\lambda) = \vartheta_n(\lambda) + m_+(\lambda)\varphi_n(\lambda)$ or of $m_+(\lambda)$ on Λ_1 (respectively on Λ_2).

From (2.5) it follows that if $\mu_n \neq \lambda_n^\pm$, $n = 1, \dots, q-1$, then one of the following two cases holds true:

(i) m_+ has simple pole at μ_n, m_- is regular and μ_n is the bound state,

(ii) m_- has simple pole at μ_n, m_+ is regular and μ_n is the antibound state.

Now suppose that either $\mu_n = \lambda_n^-$, $\lambda_0 = \mu_n + \epsilon$ or $\mu_n = \lambda_n^+$, $\lambda_0 = \mu_n - \epsilon$, $\epsilon > 0$. Then

$$m_+(\lambda_0) = \frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1), \ \epsilon \to 0, \ c \neq 0. \tag{2.6}$$

Moreover, for $n \neq 0, q$, $\psi_n^+(\lambda_0) = \vartheta_n(\mu_n) + \left(\frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1)\right)\varphi_n(\mu_n)$, the function $(\mathscr{R}_n^0(.))^2$ has a pole at μ_n for almost all $n \in \mathbb{N}$ and μ_n is the virtual state. ■

We have also

$$m_\pm = \frac{\xi_\pm - \vartheta_q}{\varphi_q}.$$

Moreover, $\mu_j \in \gamma_j$ is the antibound state iff $\xi_+(\mu_j) = \vartheta_q(\mu_j)$ and $\mu_j \in \gamma_j$ is the bound state iff $\xi_-(\mu_j) = \vartheta_q(\mu_j)$. Note that on each γ_j^+, $j = 0, 1, \dots, q$, $m_\pm$ are real functions.

Quasi-momentum map and Riemann surface $\mathcal{Z}$.

We construct the conformal mapping of the Riemann surface onto the plan with "radial cuts" $\mathcal{Z}$. Our definition corrects the similar construction in [BE] and [EMT], where there was a mistake.

We suppose that all gaps are open: $\lambda_j^- < \lambda_j^+$, $j = 1, \dots, q-1$.

Introduce a domain $\mathbb{C} \setminus \cup_0^q \overline{\gamma}_j$ and a quasi-momentum domain $\mathbb{K}$ by

$$\mathbb{K} = \{\varkappa \in \mathbb{C} : -\pi \leqslant \operatorname{Re} \varkappa \leqslant 0\} \setminus \cup_1^{q-1} \overline{\Gamma}_j, \ \ \Gamma_j = \left(-\frac{\pi j + i h_j}{q}, -\frac{\pi j - i h_j}{q} \right).$$

Here $h_j \geqslant 0$ is defined by the equation $\cosh h_j = (-1)^{j-q} \Delta(\alpha_j)$ and α_j is a zero of $\Delta'(\lambda)$ in the "gap" $[\lambda_j^-, \lambda_j^+]$. For each periodic Jacobi operator there exists a unique conformal mapping $\varkappa : \mathbb{C} \setminus \cup_0^q \overline{\gamma}_j \to \mathbb{K}$ such that the following identities and asymptotics hold true:

$$\cos q\varkappa(\lambda) = \Delta(\lambda), \quad \lambda \in \mathbb{C} \setminus \cup_0^q \overline{\gamma}_j, \qquad \text{and} \qquad \varkappa(it) \to \pm i\infty \quad \text{as} \ \ t \to \pm\infty. \tag{2.7}$$

The quasi-momentum $\varkappa$ maps the half plane $\mathbb{C}_\pm = \{\lambda \in \mathbb{C};\ \pm \operatorname{Im} \lambda > 0\}$ onto the half-strip $\mathbb{K}_\pm = \mathbb{K} \cap \mathbb{C}_\pm$ and $\sigma_{\rm ac}(J^0) = \{\lambda \in \mathbb{R};\ \operatorname{Im} \varkappa(\lambda) = 0\}$.

Define the two strips $\mathbb{K}_S$ and $\mathcal{K}$ by

$$\mathbb{K}_S = -\mathbb{K} \qquad \text{and} \qquad \mathcal{K} = \mathbb{K}_S \cup \mathbb{K} \subset \{\varkappa \in \mathbb{C} : \operatorname{Re} \varkappa \in [-\pi, \pi]\}.$$

The function $\varkappa$ has an analytic continuation from $\Lambda_1 \cap \mathbb{C}_+$ into $\Lambda_1 \cap \mathbb{C}_-$ through the infinite gaps $\gamma_q = (\lambda_q^-, \infty)$ by the symmetry and satisfies:

1) $\varkappa$ is a conformal mapping $\varkappa : \Lambda_1 \to \mathcal{K}_+ = \mathcal{K} \cap \mathbb{C}_+$, where we identify the boundaries $\{\varkappa = \pi + it, t > 0\}$ and $\{\varkappa = -\pi + it, t > 0\}$.

2) $\varkappa : \Lambda_2 \to \mathcal{K}_- = \mathcal{K} \cap \mathbb{C}_-$ is a conformal mapping, where we identify the boundaries $\{\varkappa = \pi - it, t > 0\}$ and $\{\varkappa = -\pi - it, t > 0\}$.

3) Thus $\varkappa : \Lambda \to \mathcal{K}$ is a conformal mapping.

Consider the function $z = e^{i\varkappa(\lambda)}$, $\lambda \in \Lambda$. The function $z(\lambda)$, $\lambda \in \Lambda$, is a conformal mapping $z : \Lambda \to \mathcal{Z} = \mathbb{C} \setminus \cup \overline{g}_j$, where the radial cut g_j is given by

$$g_j = (e^{-\frac{h_j}{q} + i\frac{\pi j}{q}}, e^{\frac{h_j}{q} + i\frac{\pi j}{q}}), \qquad j = \pm 1, ..., \pm(q-1).$$

The function $z(\lambda)$, $\lambda \in \Lambda$, maps the first sheet Λ_1 into the "disk" $\mathcal{Z}_1 = \mathcal{Z} \cap \mathbb{D}_1$, $\mathbb{D}_1 = \{z \in \mathbb{C} : \ |z| < 1\}$, and $z(\cdot)$ maps the second sheet Λ_2 into the domain $\mathcal{Z}_2 = \mathcal{Z} \setminus \mathbb{D}_1$. In fact, we obtain the parametrization of the two-sheeted Riemann surface Λ by the "plane" $\mathcal{Z}$. Thus below we call $\mathcal{Z}_1$ also the "physical sheet" and $\mathcal{Z}_2$ also the "non-physical sheet".

Note that if all $a_n^0 = 1, b_n^0 = 0$, then we have $\lambda = \frac{1}{2}(z + \frac{1}{z})$. This function $\lambda(z)$ is a conformal mapping from the disk $\mathbb{D}_1$ onto the cut domain $\mathbb{C} \setminus [-2, 2]$.

Now, the functions $\psi^\pm(\lambda)$ can be considered as functions of $z \in \mathcal{Z}$. The functions $\psi_n^\pm(z) \equiv \psi_n^\pm(\lambda(z))$ are meromorphic in $\mathcal{Z}$ with the only possible singularities at the images of the Dirichlet eigenvalues $z(\mu_j) \in \mathcal{Z}$ and at 0. More precisely,
1) $\psi_n^\pm$ are analytic in $\mathcal{Z} \setminus (\{z(\mu_j)\}_{j=1}^{q-1} \cup \{0\})$ and continuous up to $\partial\mathcal{Z} \setminus \{z(\mu_j)\}_{j=1}^{q-1}$.
2) $\psi_n^\pm(z)$ has a simple pole at $z(\mu_j) \in \mathcal{Z}$ if μ_j is a pole of $m_\pm$, no pole if μ_j is not a singularity of $m_\pm$ (not a square root singularity if μ_j coincides with the band edge) and if μ_j coincides with the band edge: $\mu_j = \lambda_j^\sigma$, $\sigma = +$ or $\sigma = -$, $j = 1, \ldots, q-1$, then

$$\psi_n^\pm(z) = \pm\sigma(-1)^{q-j} \frac{iC(n)}{z - z(\lambda_j^\sigma)} + \mathcal{O}(1), \quad \lambda \in [\lambda_{j-1}^+, \lambda_j^-], \tag{2.8}$$

for some constant $C(n) \in \mathbb{R}$. Note that the sign comes from the analytic continuation of the square root $\Omega(\lambda)$ using the definition (1.14).
3) The following identities hold true:

$$\psi_n^\pm(\overline{z}) = \psi_n^\pm(z^{-1}) = \psi_n^\mp(z) = \overline{\psi_n^\pm(z)} \text{ as } |z| = 1. \tag{2.9}$$

4) The following asymptotics hold true:

$$\psi_n^\pm(z) = (-1)^n \Big(\prod_{j=0}^{n-1} {}^*a_j\Big)^{\pm 1} z^{\pm n} \Big(1 + \mathcal{O}(z)\Big) \qquad \text{as} \qquad z \to 0.$$

We collect below some properties of the quasi-momentum $\varkappa$ on the gaps.

On each $\gamma_j^+, j=0,1,\dots,q$, the quasi-momentum $\varkappa(\lambda)$ has constant real part and positive $\operatorname{Im}\varkappa$:

$$\operatorname{Re}\varkappa|_{\gamma_j^+} = -\frac{q-j}{q}\pi, \qquad \varkappa(\lambda_j^-) = \varkappa(\lambda_j^+) = -\frac{q-j}{q}\pi, \qquad \operatorname{Im}\varkappa|_{\gamma_j^+} > 0.$$

Moreover, as λ increases from λ_j^- to α_j the imaginary part $\operatorname{Im}\varkappa \equiv h(\lambda)$ is monotonically increasing from 0 to h_j and as λ increases from α_j to λ_j^- the imaginary part $\operatorname{Im}\varkappa \equiv h(\lambda+i0)$ is monotonically decreasing from h_j to 0. Then

$$\frac{1}{2}\varphi_q(\lambda)(m_+(\lambda) - m_-(\lambda)) = \sqrt{\Delta^2(\lambda)-1} = i\sin q\varkappa(\lambda) = -(-1)^{q-j}\sinh qh(\lambda+i0), \quad (2.10)$$

where $\sinh qh = -2^{-1}(z^q - z^{-q}) > 0$.

2.2 The perturbed Jacobi operator, Jost functions.

We consider the operator $J = J^0 + V$ given by (1.6). Recall that $f_n^\pm$ are solutions to the equation

$$a_{n-1}y_{n-1} + a_n y_{n+1} + b_n y_n = \lambda y_n, \quad \lambda \in \Lambda_1, \quad (2.11)$$

satisfying

$$f_n^\pm = \psi_n^\pm, \qquad \text{for all } \ n \geqslant p+1. \quad (2.12)$$

Recall that $a_n = a_n^0 + u_n, \quad b_n = b_n^0 + v_n$. Equation (2.11) has unique solutions ϑ_n^+, φ_n^+ such that

$$\vartheta_n^+ = \vartheta_n, \qquad \varphi_n^+ = \varphi_n, \qquad \text{for all } \ n \geqslant p+1.$$

The functions $\vartheta_n^+(\cdot)$, $\varphi_n^+(\cdot)$ are polynomials. The functions $f_n^\pm$ have the form

$$f_n^\pm = \vartheta_n^+ + m^\pm \varphi_n^+ \quad (2.13)$$

and satisfy $\overline{f_n^\pm}(\overline{\lambda}) = f_n^\mp(\lambda)$, $\lambda \in \Gamma$.

Lemma 2.1. *The zeros of the polynomials ϑ_0^+ and φ_0^+ are disjoint.*

Proof. Assume that $\vartheta_0^+(\lambda_0) = \varphi_0^+(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}$. Then $\vartheta_n^+(\lambda_0) = a\varphi_n^+(\lambda_0)$ for all $n \geqslant 1$ and some $a \neq 0$. Then (1.18) gives $\vartheta_n(\lambda_0) = a\varphi_n(\lambda_0)$ for all $n > p$ and thus $\vartheta_n(\lambda_0) = a\varphi_n(\lambda_0)$ for all $n > 1$ and the Wronskian $\{\vartheta(\lambda_0), \varphi(\lambda_0)\} = 0$. We have a contradiction, since $\{\vartheta(\lambda_0), \varphi(\lambda_0)\} = 1$. ■

By Definition 1 a state is a singularity of the resolvent. The kernel of the resolvent of J is given by

$$R(m,n) = \langle e_m, (J-\lambda)^{-1} e_n\rangle = -\frac{\Phi_m f_n^+}{\{\Phi, f^+\}} = \frac{\Phi_m \mathscr{R}_n(\lambda)}{a_0}, \ m < n,$$

$$\mathscr{R}_n(\lambda) = \frac{f_n^+(\lambda)}{f_0^+(\lambda)}.$$

Here $e_n = (\delta_{n,j})_1^\infty$ is the unit vector in ℓ^2, and $\Phi = (\Phi_n)_0^\infty$ is a solution of the equation (2.11) under the condition $\Phi_0 = 0$, $\Phi_1 = 1$, and note that $\{\Phi, f^+\} = -a_0 f_0^+$. Each function $\Phi_n(\lambda)$, is polynomial in λ. The function $R(n,m)$ is meromorphic on Λ for each $n, m \in \mathbb{N}$. Then the singularities of $R(n,m)$ are given by the singularities of $\mathscr{R}_n(\lambda)$. We have

Lemma 2.2. *1) A real number $\lambda_0 \in \gamma_k^+$, $k = 0, 1, \dots, q$ is a bound state, if the function $\mathscr{R}_n(\lambda)$ has a pole at λ_0 for some $n \in \mathbb{N}$. Recall (see Introduction, before Definition 1) that the bound states are simple.*
2) A number $\lambda_0 \in \Lambda_2$, is a resonance, if the function $\mathscr{R}_n(\lambda)$ has a pole at λ_0 for some $n \in \mathbb{N}$. The multiplicity of the resonance is the multiplicity of the pole.
3) A real number $\lambda_0 = \lambda_k^\pm$, $k = 0, \dots, q$, is a virtual state if $\mathscr{R}_n^2(\lambda)$ or $\mathscr{R}_n(\lambda)$ has a pole at λ_0 for some $n \in \mathbb{Z}_+$.

Proof of Theorem 1.1 i) We start with the case $\lambda_0 \notin \sigma_{\rm st}(J^0)$.

Let $\Omega(\lambda_0) \neq 0$. Then f_n^+, $n \in \mathbb{N}$, is analytic at $\lambda_0 \in \Lambda$. Then $\mathscr{R}_n(\lambda)$ has a pole at λ_0 iff $f_0^+(\lambda_0) = 0$.

Let now $\Omega(\lambda_0) = 0$. Using (2.2) we get $m^\pm(\lambda) = m^\pm(\lambda_0) + c\sqrt{\epsilon} + \mathcal{O}(\epsilon)$, $\lambda - \lambda_0 = \epsilon \to 0$, and $c \neq 0$. We distinguish between two cases.
a) Firstly, let $\varphi_0^+(\widetilde{\lambda}_0) \neq 0$. Then identity $f_0^+(\lambda_0) = \vartheta_0^+(\lambda_0) + m^+\varphi_0^+(\lambda_0) = 0$ implies (2.18)

$$f_0^+(\lambda) = \varphi_0^+(\widetilde{\lambda}_0) c\sqrt{\epsilon} + \mathcal{O}(\epsilon), \ \mathscr{R}_n(\lambda) = \frac{f_n^+(\lambda)}{\varphi_0^+(\widetilde{\lambda}_0)c\sqrt{\epsilon}}(1 + \mathcal{O}(\sqrt{\epsilon})), \ c\varphi_0^+(\widetilde{\lambda}_0) \neq 0.$$

Then λ_0 is a virtual state of J.
b) Secondly, if $\varphi_0^+(\widetilde{\lambda}_0) = 0$, then we obtain $\vartheta_0^+(\widetilde{\lambda}_0) \neq 0$ by Lemma 2.1 and $f_0^+(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$. Then λ_0 is not a singularity of the resolvent.

Now we consider the case $\lambda_0 \in \sigma_{\rm st}(J^0)$. Then $\varphi_q(\widetilde{\lambda}_0) = 0$.
Suppose firstly that $\Omega(\lambda_0) \neq 0$. Then λ_0 is a pole of m_+ and therefore λ_0 is a pole of the Jost solution $f_n^+(\lambda) = \vartheta_n^+ + m_+\varphi_n^+$ on either Λ_1 or Λ_2 for all $n \in \mathbb{N}$ such that $\varphi_n^+(\widetilde{\lambda}_0) \neq 0$. Then using Lemma 2.1 we get that if $\varphi_0^+(\widetilde{\lambda}_0) = 0$ then $f_0^+(\lambda_0) \neq 0$ and λ_0 is a pole of

$$\mathscr{R}_n(\lambda) = \frac{f_n^+(\lambda)}{f_0^+(\lambda)} = \frac{\vartheta_n^+ + m_+\varphi_n^+}{\vartheta_0^+ + m_+\varphi_0^+}, \quad n \in N,$$

iff $\varphi_0^+(\widetilde{\lambda}_0) = 0$. Moreover, λ_0 is a simple state (as a pole of m^+).

Suppose now that $\lambda_0 \in \sigma_{\rm st}(J^0)$ and $\Omega(\lambda_0) = 0$.
Then we have (2.6):

$$m^+(\lambda) = \frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1), \ \lambda - \lambda_0 = \epsilon \to 0, \ c \neq 0.$$

We distinguish between two cases.
a) Firstly, let $\varphi_0^+(\widetilde{\lambda}_0) \neq 0$. Then identity $f_0^+(\lambda_0) = \vartheta_0^+(\lambda_0) + m^+(\lambda_0)\varphi_0^+(\lambda_0) = 0$ implies

$$f_0^+(\lambda) = \frac{\varphi_0^+(\widetilde{\lambda}_0)c}{\sqrt{\epsilon}} + \mathcal{O}(1), \qquad \frac{f_n^+(\lambda)}{f_0^+(\lambda)} = \frac{\vartheta_n^+(\widetilde{\lambda}) + \left(\frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1)\right)\varphi_n^+(\widetilde{\lambda})}{\frac{\varphi_0^+(\widetilde{\lambda}_0)c}{\sqrt{\epsilon}} + \mathcal{O}(1)} = \frac{1 + \mathcal{O}(\sqrt{\epsilon})}{\varphi_0^+(\widetilde{\lambda}_0)},$$

and each function $\mathscr{R}_n(.)$, $n \in \mathbb{N}$, does not have singularity at λ_0.
b) Secondly, let $\varphi_0^+(\widetilde{\lambda}_0) = 0$. Then $f_0^+(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$ by Lemma 2.1. Moreover, we obtain $f_n^+(\lambda) = \vartheta_n^+(\widetilde{\lambda}) + \left(\frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1)\right) \varphi_n^+(\widetilde{\lambda})$, and the function $(\mathscr{R}_n(.))^2$, $n \in \mathbb{N}$, has simple pole at λ_0.

ii) Suppose $\lambda_1 \in \Lambda_1$ is a bound state of J and $\lambda_1 \notin \sigma_{\rm st}(J^0)$. Then by i) we have $f_0^+(\lambda_1) = 0$ and as $\{f^+, f^-\} \neq 0$ we have $f_0^-(\lambda_1) \neq 0$ (by the argument similar to Lemma 2.1). The last identity is equivalent to $f_0^+(\lambda_2) \neq 0$ for $\lambda_2 \in \Lambda_2$ such that $\widetilde{\lambda}_2 = \widetilde{\lambda}_1$.

iii) In i) it was shown that if $\lambda_0 \in \sigma_{\rm st}(J^0)$ then $f_0^+(\lambda_0) \neq 0$. So it is enough to consider the case $\lambda_0 \in \Lambda$ is a zero of f_0^+ and $\lambda_0 \notin \sigma_{\rm st}(J^0)$. If $\varphi_0^+(\widetilde{\lambda}_0) = 0$ then $f_0^+(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$ as in ii) which is a contradiction.

■

Define the function

$$F_n(\lambda) = \varphi_q(\lambda) f_n^+(\lambda) f_n^-(\lambda), \quad \lambda \in \Lambda_1. \tag{2.14}$$

Note that $F_0 = F$ defined previously in (1.22). Using (2.13) and (1.16), (2.4), (2.5) we get

$$F_n = \varphi_q(\vartheta_n^+)^2 + 2\phi\vartheta_n^+\varphi_n^+ - \vartheta_{q+1}(\varphi_n^+)^2, \quad n \geqslant 0. \tag{2.15}$$

The following Lemma is proven in Section 4.

Lemma 2.3. *Let $\nu \in \{2p, 2p-1\}$. Each function $F_n(\lambda) = \varphi_q(\lambda) f_n^+(\lambda) f_n^-(\lambda)$, $n \geqslant 0$, is a polynomial and satisfy*

$$F_n(\lambda) = -a_0^0 \lambda^{\kappa - 2n}\left(c_3(n) + \mathcal{O}(\lambda^{-1})\right), \qquad \kappa = \nu + q - 1, \qquad \lambda \to \infty, \tag{2.16}$$

$$c_3(n) = c_1(n)c_2(n), \quad c_1(n) = \frac{1}{\prod_{j=n}^p a_j}, \quad c_2(n) = \begin{cases} c_1(n)u_p(a_p^0 + a_p) & \text{if} \quad \nu = 2p, \\ c_1(a_p^0)^2 v_p & \text{if} \quad \nu = 2p-1. \end{cases} \tag{2.17}$$

Remark. It follows that the function $F_n(\lambda) = \varphi_q(\lambda) f_n^+(\lambda) f_n^-(\lambda)$ is polynomial of degree $2(p-n) + q - 1$ (if $u_p \neq 0$) or $2(p-n) + q - 2$ (if $u_p = 0$, $v_p \neq 0$). From the asymptotics (4.2), (4.3) collected in Section 4, we get the sign of $F = F_0$ as $\lambda \to \infty$:

$$\operatorname{sign} F(\lambda) = \begin{cases} -\operatorname{sign} u_p & if \ u_p \neq 0 \\ -\operatorname{sign}(v_p) & if \ a_p^0 \neq a_p \end{cases} \quad as \ \lambda \to \infty,$$

$$\operatorname{sign} F(\lambda) = \begin{cases} (-1)^{2p+q-2} \operatorname{sign} u_p & if \ u_p^0 \neq 0 \\ -(-1)^{2p+q-2} \operatorname{sign}(v_p) & if \ u_p^0 = 0, v_p \neq 0 \end{cases} \quad as \ \lambda \to -\infty.$$

We summarize the results about the virtual states $\sigma_{\rm vs}(J)$ obtained in the proof of Theorem 1.1 in the following Lemma.

Lemma 2.4 (Virtual states)**.** *Let $\lambda_0 = \lambda_k^{\pm}$ for some $k = 0, \dots, q-1$. If $\lambda_0 = \lambda_k^+$ then put $\lambda = \lambda_0 - \epsilon$. If $\lambda_0 = \lambda_k^-$, then put $\lambda = \lambda_0 + \epsilon$. Here $\epsilon > 0$ is small enough.*
i) Let $\lambda_0 \notin \sigma_{\rm st}(J^0)$ and $f_0^+(\lambda_0) = 0$. Then $\widetilde{\lambda}_0$ is a simple zero of F, λ_0 is virtual state of J and

$$f_0^+(\lambda) = \varphi_0^+(\widetilde{\lambda}_0)c\sqrt{\epsilon} + \mathcal{O}(\epsilon),\ \mathscr{R}_n(\lambda) = \frac{f_n^+(\lambda)}{\varphi_0^+(\widetilde{\lambda}_0)c\sqrt{\epsilon}}(1 + \mathcal{O}(\sqrt{\epsilon})),\ c\varphi_0^+(\widetilde{\lambda}_0) \neq 0. \tag{2.18}$$

ii) Let $\lambda_0 \in \sigma_{\rm st}(J^0)$ and $\varphi_0^+(\widetilde{\lambda}_0) \neq 0$. Then $F(\widetilde{\lambda}_0) \neq 0$ and each $\mathscr{R}_n(.)$, $n \in \mathbb{N}$, does not have singularity at λ_0 and λ_0 is not a virtual state of J.
iii) Let $\lambda_0 \in \sigma_{\rm st}(J^0)$ and $\varphi_0^+(\widetilde{\lambda}_0) = 0$. Then λ_0 is virtual state of J, $f_0^{\pm}(\lambda_0) \neq 0$, $\widetilde{\lambda}_0$ is simple zero of F, and each $(\mathscr{R}_n(.))^2$, $n \in \mathbb{N}$, has pole at λ_0.

In the next Lemma we show identification of the states of J and zeros of polynomial F.

Lemma 2.5. *The projection $\widetilde{\ }$: $\Lambda \mapsto \mathbb{C}$ of the set of states of J on Λ coincides with the set of zeros of F on the complex plane $\mathbb{C}$:*

$$\widetilde{\sigma}_{\rm st}\,(J) = \text{Zeros}\,(F).$$

Moreover, the multiplicities of bound states and resonances are equal to the multiplicities of zeros of F. All bound states are simple. The virtual state is a simple zero of F.

Proof: First we observe that $f_0^+(\lambda)$ is analytic on $\Lambda \setminus \sigma_{\rm st}\,(J^0)$.

By Theorem1.1 a point $\lambda_0 \in \gamma_k^+$, $\lambda_0 \notin \sigma_{\rm st}(J^0)$, is a bound state iff $f_0^+(\lambda_0) = 0$. Then $f_0^-(\lambda_0) \neq 0$ as the Wronskian $\{f_0^+, f_0^-\}(\lambda_0) \neq 0$. Moreover, it follows that $\widetilde{\lambda}_0$ is zero of $F(\lambda)$ with the same multiplicity (one).

A point $\lambda_0 \in \Lambda_2$, $\lambda_0 \notin \sigma_{\rm st}(J^0)$, $\Omega(\lambda_0) \neq 0$, is a resonance iff $f_0^+(\lambda_0) = 0$ which is equivalent to $f_0^-(\lambda_1) = 0$, where λ_1 is the same number as λ_0 but on the physical sheet. Then it follows that $F(\widetilde{\lambda}_0) = 0$ with the same multiplicity.

If $F(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{R}$, $\lambda_0 \notin \sigma_{\rm st}(J^0)$, $\Omega(\lambda_0) \neq 0$, then it is clear that there is either a bound state $\lambda_0^1 \in \Lambda_1$ with $\widetilde{\lambda}_0^1 = \lambda_0$ or an antibound $\lambda_0^2 \in \Lambda_2$ state with $\widetilde{\lambda}_0^2 = \lambda_0$ with the same multiplicity as λ_0.

If $F(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, then necessarily $f_0^+(\lambda_0^2) = 0$ at $\lambda_0^2 \in \Lambda_2$, with $\widetilde{\lambda}_0^2 = \lambda_0$, and λ_0^2 is the complex resonance with the same multiplicity as λ_0.

Consider now a point $\lambda_0 \in \gamma_1^+$ or $\lambda_0 \in \gamma_1^-$ such that $\lambda_0 \in \sigma_{\rm st}(J^0)$ and $\varphi_n^+(\widetilde{\lambda}_0) \neq 0$ for some $n > 0$. Then m_+ has a pole at λ_0, and $f_n^+(\lambda)$ has a simple pole at λ_0. Then λ_0 is a pole of

$$\mathscr{R}_n(\lambda) = \frac{f_n^+(\lambda)}{f_0^+(\lambda)} = \frac{\vartheta_n^+ + m_+\varphi_n^+}{\vartheta_0^+ + m_+\varphi_0^+}$$

iff $\varphi_0^+(\widetilde{\lambda}_0) = 0$, as by Lemma 2.1 in this case $\vartheta_0^+(\widetilde{\lambda}_0) \neq 0$.

Now using the identity $F_0 = \varphi_q f_0^+(\lambda) f_0^-(\lambda) = \varphi_q(\vartheta_0^+)^2 + (\varphi_{q+1} - \vartheta_q)\vartheta_0^+\varphi_0^+ - \vartheta_{q+1}(\varphi_0^+)^2$ we get that if $\varphi_q(\widetilde{\lambda}) = \varphi_0^+(\widetilde{\lambda}) = 0$, then necessarily $\widetilde{\lambda}$ is a simple zero of F_0 and $f_0^{\pm}(\lambda) \neq 0$.

The other statements of Lemma follows similarly as in the proof of Theorem 1.1 ■

Let $M_\pm \in \mathbb{C}$ denote (the projection of) the set of poles of $m_\pm$. Let $M_{\rm e}$ denote the set of square root singularities of $m_\pm$ if $\mu_k = \lambda_k^+$ or $\mu_k = \lambda_k^-$, $k = 1, \ldots, q-1$. Note that $M_+ \cap M_- = \emptyset$. We put

$$D^\pm = \prod_{\mu_k \in M_\pm} (\widetilde{\lambda} - \mu_k), \qquad D^{\rm e} = \prod_{\mu_k \in M_{\rm e}} \sqrt{\widetilde{\lambda} - \mu_k},$$

where $\widetilde{\ }: \ \Lambda \mapsto \mathbb{C}$ is the natural projection introduced in (1.12). Let $\mu_\pm = \sharp\,(M_\pm)$, $\mu_{\rm e} = \sharp\,(M_{\rm e})$, be the number of elements in the respective sets. If all gaps are open ($\lambda_n^- < \lambda_n^+$, $n = 1, \ldots, q$) then we have $\mu_+ + \mu_- + \mu_{\rm e} = q-1$ and $\varphi_q = a_0^0 (D^{\rm e})^2 D^+ D^-$. We mark with $\hat{\ }$ the modified (regularized) quantities: $\hat{\psi}^\pm = D^{\rm e} D^\pm \psi^\pm$, $\hat{f}^\pm = D^{\rm e} D^\pm f^\pm$. Now $\hat{\psi}^\pm$, $\hat{f}^\pm$ are analytic in Λ_1.

In the next Lemma we prove the crucial property for the function $F \equiv F_0 = \varphi_q f_0^+ f_0^- = a_0^0 \hat{f}_0^+ \hat{f}_0^-$. Recall that $\{\phi_n, \psi_n\} = a_n(\phi_n \psi_{n+1} - \phi_{n+1}\psi_n\}$ denotes the Wronskian. Let as before $\dot{y} = \partial_\lambda y = \partial y/\partial\lambda$ and define the difference derivative

$$\partial_n f(n) = f(n+1) - f(n).$$

Lemma 2.6. *i) Any solution y_n of (1.6) satisfies*

$$\partial_n \{\dot{y}, y\}_n = -(y_{n+1})^2, \ \forall n \geqslant 0. \tag{2.19}$$

ii) Suppose that $\lambda_1 \in \gamma_k^+$, for $k = 0, 1, \ldots, q$ and $\hat{f}_0^+(\lambda_1) = 0$, i.e. λ_1 is an eigenvalue of J with the eigenfunction $y_n = \hat{f}_n^+(\lambda_1)$. Then

$$m_1 := \sum_{k=0}^\infty \left(\hat{f}_k^+(\lambda_1)\right)^2 = a_0 \left(\frac{\partial}{\partial\lambda}\hat{f}_0^+\right)\hat{f}_1^+ > 0 \quad at\ \lambda = \lambda_1; \tag{2.20}$$

$$\{\hat{f}^+, \hat{f}^-\}_n = \varphi_q(m_- - m_+); \tag{2.21}$$

$$m_1 = \frac{\dot{F}(\lambda_1)}{a_0^0(\hat{f}_0^-(\lambda_1))^2} \cdot (-1)^{q-k+1} 2\sinh qh(\lambda_1) = \frac{(\partial_\lambda \hat{f}_0^+)(\lambda_1)}{\hat{f}_0^-(\lambda_1)} \cdot (-1)^{q-k+1} 2\sinh qh(\lambda_1) > 0, \tag{2.22}$$

where $h(\lambda_1) = \operatorname{Im} \varkappa(\lambda_1) > 0$. Thus $(-1)^{q-k}\dot{F}(\lambda_1) < 0$ and the function F has simple zeros at all bound states of J for which $\varphi_q \neq 0$. If $\lambda_0 = \mu_k$ is an antibound state then necessarily it is simple and $(-1)^{q-k}\dot{F}(\lambda_0) > 0$.

Remark. As by Lemma 2.5 the zeros of F coincide with the projections of states of J to $\mathbb{C}$ then Lemma 2.6 implies that between any two (projections of) eigenvalues $\lambda_1, \lambda_3 \in \gamma_k$ (not separated by a band of the absolute continuous spectrum) there is at least one (projection of) real resonance (antibound state) λ_2 such that $(-1)^{q-k}\dot{F}(\lambda_2) > 0$.

Proof. i) Using $y_{n+2} = \frac{1}{a_{n+1}}((\lambda - b_{n+1})y_{n+1} - a_n y_n)$, we get

$$\partial_n \left[a_n(\dot{y}_n)y_{n+1} - a_n(\dot{y}_{n+1})y_n\right] = -(y_{n+1})^2,$$

which yields (2.19).

ii) Note the following "telescopic" sum $\sum_{k=n}^{m} \partial y_k = y_{m+1} - y_n$. We put $n = 0$ and get from (2.19)

$$\{\dot{y}, y\}_{m+1} - a_0 \left[(\dot{y}_0)\, y_1 - (\dot{y}_1)\, y_0\right] = -\sum_{k=0}^{m} y_{k+1}^2.$$

We put $\lambda = \lambda_1$ and $y = \hat{f}^+(\lambda_1)$. Then, using that the eigenfunction $\hat{f}^+(\lambda_1) \in \ell^2(\mathbb{N})$ and $\hat{f}_m^+ \to 0$ as $m \to \infty$, we get that the first term in the left hand side goes to zero. As λ_1 is an eigenvalue, then we have $\hat{f}_0^+(\lambda_1) = 0$ and we get

$$-a_0 \left(\frac{\partial}{\partial \lambda} \hat{f}_0^+\right) \hat{f}_1^+ = -\sum_{k=0}^{\infty} (\hat{f}_{k+1}^+)^2 \text{ at } \lambda = \lambda_1.$$

Finally we get (2.20) using that $\hat{f}^+(\lambda_1) \in \mathbb{R}$.

Next formula (2.21) follows from const $= \{f_n^+, f_n^-\} = \{\psi_n^+, \psi_n^-\} = \{\psi_0^+, \psi_0^-\} = a_0^0(m_- - m_+)$.

Putting $n = 0$ we get also $\{f_n^+, f_n^-\} = -a_0 f_1^+(\lambda_1) f_0^-(\lambda_1)$ using again $f_0^+(\lambda_1) = 0$. Together with (2.21) and definitions of $m_\pm$ it implies

$$\begin{aligned} \hat{f}_1^+(\lambda_1)\hat{f}_0^-(\lambda_1) &= \frac{1}{a_0^0}\varphi_q f_1^+(\lambda_1) f_0^-(\lambda_1) = \frac{\varphi_q}{a_0}(m_+ - m_-) = \frac{i2\sin q\varkappa(\lambda_1)}{a_0} \\ \Rightarrow\ \hat{f}_1^+(\lambda_1) &= \frac{i2\sin q\varkappa(\lambda_1)}{a_0 \hat{f}_0^-(\lambda_1)}. \end{aligned} \tag{2.23}$$

Recall that $F(\lambda) = a_0^0 \hat{f}_0^+ \hat{f}_0^-$. Taking the derivative of F with respect to λ, we get $\dot{F}(\lambda_1) = a_0^0(\partial_\lambda \hat{f}_0^+)(\lambda_1)\hat{f}_0^-(\lambda_1)$, wherefrom it follows

$$(\partial_\lambda \hat{f}_0^+)(\lambda_1) = \frac{\dot{F}(\lambda_1)}{a_0^0 \hat{f}_0^-(\lambda_1)}. \tag{2.24}$$

Inserting (2.23) and (2.24) in (2.20): $m_1 = \sum_{k=0}^{\infty} \left|\hat{f}_k^+(\lambda_1)\right|^2 = a_0(\partial_\lambda \hat{f}_0^+)(\lambda_1)\hat{f}_1^+(\lambda_1)$, we get

$$m_1 = \dot{F}(\lambda_1) \cdot \frac{i2\sin q\varkappa(\lambda_1)}{a_0^0(\hat{f}_0^-(\lambda_1))^2} > 0.$$

For $\lambda_1 \in \gamma_k^+$ for $k = 0, 1 \ldots, q$, $\operatorname{Im} \varkappa(\lambda_1) = h(\lambda_1) > 0$. Then by (2.10) $i \sin q\kappa(\lambda_1) = -(-1)^{q-k} \sinh qh(\lambda_1 + i0)$, which implies (2.22).

■

Lemma 2.7. *i) The following identity holds true*

$$F = \varphi_q \left(\vartheta_0^+ + \frac{\phi}{\varphi_q}\varphi_0^+\right)^2 + \frac{1 - \Delta^2}{\varphi_q}(\varphi_0^+)^2. \tag{2.25}$$

Moreover, $F(\lambda)\neq 0$, *for any* $\lambda\in(\lambda_{n-1}^+,\lambda_n^-)$, $n=1,\dots,q$, *and* $\operatorname{sign}F|_{(\lambda_{n-1}^+,\lambda_n^-)}=\operatorname{sign}\varphi_q|_{(\lambda_{n-1}^+,\lambda_n^-)}$.

ii) If $\lambda_0\in\{\lambda_{n-1}^+,\lambda_n^-\}$ *is a virtual state, then* F *has a simple zero at* λ_0.

iii) There is always odd number $\geqslant 1$ *of states (eigenvalues, antibound or virtual state) in each finite open gap* $\gamma_n^c=\overline{\gamma}_n^-\cup\overline{\gamma}_n^+$, $n=1,\dots,q-1$.

Proof. i) Using (1.16) and (2.5) we obtain

$$F=\varphi_q\Big((\vartheta_0^+)^2+(m_++m_-)\vartheta_0^+\varphi_0^++m_+m_-(\varphi_0^+)^2\Big)=\varphi_q\Big((\vartheta_0^+)^2+\frac{2\phi}{\varphi_q}\vartheta_0^+\varphi_0^+-\frac{\vartheta_{q+1}}{\varphi_q}(\varphi_0^+)^2\Big)$$

$$=\varphi_q\Big(\vartheta_0^++\frac{\phi}{\varphi_q}\varphi_0^+\Big)^2+\frac{\phi^2-\vartheta_{q+1}\varphi_q}{\varphi_q}(\varphi_0^+)^2=\varphi_q\Big(\vartheta_0^++\frac{\phi}{\varphi_q}\varphi_0^+\Big)^2+\frac{1-\Delta^2}{\varphi_q}(\varphi_0^+)^2.$$

Now ii) and iii) follow directly from i).

■

Now **the proof of Theorem 1.2** follows from the properties of the function $F=\varphi_q f^+f^-$, stated in Lemmata 2.2–2.7.

■

In the next lemma we consider the zeros of the function $S(\lambda)-1$ which are the solutions of the equation $f_0^+(\lambda)=f_0^-(\lambda)$. Note that if $\lambda_1\in\Lambda_1$ is a zero of $S-1$ then also $\lambda_2\in\Lambda_2$ such that $\widetilde{\lambda}_2=\widetilde{\lambda}_1$ is a zero of $S-1$.

Lemma 2.8. *Let* $\lambda_0\in\Lambda$ *and* $\widetilde{\lambda}_0\in\mathbb{C}$ *denote the projection on* Λ_1.
i) Suppose that $\varphi_0^+(\widetilde{\lambda}_0)=0$ *and one of the following conditions is satisfied:*

1) $\lambda_0\notin\sigma_{\rm st}(J^0)$.
2) $\lambda_0\in\sigma_{\rm st}(J^0)$, $\Omega(\lambda_0)\neq 0$ *and* $\widetilde{\lambda}_0$ *is zero of* φ_0^+ *of multiplicity* $\geqslant 2$.
3) $\lambda_0\in\sigma_{\rm st}(J^0)$ *and* $\Omega(\lambda_0)=0$.
Then $S(\lambda_0)=1$.

ii) Suppose that $S(\lambda_0)=1$ *and one of the following conditions is satisfied:*

1) $\lambda_0\notin\sigma_{\rm st}(J^0)$ *and* $\Omega(\lambda_0)\neq 0$.
2) $\lambda_0\in\sigma_{\rm st}(J^0)$ *and* $\Omega(\lambda_0)\neq 0$.
3) $\lambda_0\in\sigma_{\rm st}(J^0)$ *and* $\Omega(\lambda_0)=0$.
Then $\varphi_0^+(\widetilde{\lambda}_0)=0$.

In the cases 1) and 3) the zeros (and their multiplicities) of φ_0^+ *and* $1-S$ *coincide.*

Proof. i) Note the identities following from (1.29)

$$1-S(\lambda_0)=\frac{f_0^+(\lambda_0)-f_0^-(\lambda_0)}{f_0^+(\lambda_0)}=\frac{2i\Omega(\lambda_0)}{\varphi_q(\widetilde{\lambda}_0)}\frac{\varphi_0^+(\widetilde{\lambda}_0)}{f_0^+(\lambda_0)}. \tag{2.26}$$

Note that $\lambda_0\in\sigma_{\rm st}(J^0)$ iff $\varphi_q(\widetilde{\lambda}_0)=0$. Assume that $\varphi_q(\widetilde{\lambda}_0)\neq 0$. Then $f_0^\pm$ are analytic at λ_0 and due to Lemma 2.1 we obtain $f_0^\pm(\lambda_0)=\vartheta_0^+(\widetilde{\lambda}_0)\neq 0$. Using this we get $S(\lambda_0)=\frac{f_0^-(\lambda_0)}{f_0^+(\lambda_0)}=1$. This is also true for $\Omega(\lambda_0)=0$.

Assume now that $\lambda_0 \in \sigma_{\rm st}(J^0)$. We distinguish between two cases.

Firstly, let $\widetilde{\lambda}_0 \in \Lambda_1$ be a zero of φ_0^+ with multiplicity $\geqslant 2$. Then $f_0^\pm(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$, since $\widetilde{\lambda}_0$ is a simple zero of φ_q. Thus $S(\lambda_0) = \frac{f_0^-(\lambda_0)}{f_0^+(\lambda_0)} = 1$.

Secondly, let $\widetilde{\lambda}_0 \in \Lambda_1$ be a simple zero of φ_0^+. Suppose $\Omega(\lambda_0) \neq 0$. As $\lambda_0 \in \sigma_{\rm st}(J^0)$, then the point $\lambda_0 \in \Lambda$ is a pole of m_+. Then m_- is analytic at λ_0 and using (1.16) we have

$$f_0^+(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) + \frac{2\phi(\widetilde{\lambda}_0)}{\dot{\varphi}_q(\widetilde{\lambda}_0)}\dot{\varphi}_0^+(\widetilde{\lambda}_0), \qquad f_0^-(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0). \tag{2.27}$$

This yields $f_0^+(\lambda_0) \neq f_0^-(\lambda_0)$, since $\frac{2\phi(\widetilde{\lambda}_0)}{\dot{\varphi}_q(\widetilde{\lambda}_0)}\dot{\varphi}_0^+(\widetilde{\lambda}_0) \neq 0$. Note that $\vartheta_0^+(\widetilde{\lambda}_0) \neq 0$. Then $S(\lambda_0) \neq 1$.

Suppose now that $\Omega(\lambda_0) = 0$. Then

$$m^\pm(\lambda) = \frac{c}{\sqrt{\epsilon}} + \mathcal{O}(1), \qquad \lambda = \lambda_0 + \epsilon, \qquad \epsilon \to 0+, \qquad c \neq 0,$$

and $f_0^\pm(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$ which implies that $S(\lambda_0) = 1$.

ii) Let $S(\lambda_0) = 1$. We use (1.29)

$$\varphi_0^+ = \frac{\varphi_q}{2i\Omega}\left(f_0^+ - f_0^-\right) = \frac{\varphi_q}{2i\Omega} f_0^+ \left(1 - S\right).$$

If $\Omega(\lambda_0) \neq 0$ and $\varphi_q(\widetilde{\lambda}_0) \neq 0$, then $f_0^\pm$ are bounded near λ_0 and we have $\varphi_0^+(\widetilde{\lambda}_0) = 0$.

If $\Omega(\lambda_0) \neq 0$ and $\lambda_0 \in \sigma_{\rm st}(J^0)$, then $\widetilde{\lambda}_0$ is the zero of φ_0^+, and from (2.27) it follows that the multiplicity of $\widetilde{\lambda}_0$ is $\geqslant 2$.

If $\Omega(\lambda_0) = 0$ and $\varphi_q(\widetilde{\lambda}_0) \neq 0$, then $\varphi_0^+(\widetilde{\lambda}_0) \neq 0$.

If $\Omega(\lambda_0) = 0$ and $\lambda_0 \in \sigma_{\rm st}(J^0)$, then we get $f_0^+(\lambda_0) = \vartheta_0^+(\widetilde{\lambda}_0) \neq 0$ and $\varphi_0^+(\widetilde{\lambda}_0) = 0$ as $\Omega(\lambda) = c\sqrt{\epsilon} + \mathcal{O}(\epsilon)$ as $\lambda - \lambda_0 = \epsilon \to 0+$. ∎

3 Inverse problem

3.1 Preliminaries

In this section we collect some properties of the Jost solutions needed for the proof of the inverse results. The first lemma states that that the Jost solutions $f^\pm$ inherit the properties of $\psi^\pm$.

Lemma 3.1. *1) Each $f_n^\pm, n \geqslant 0$, is analytic in $\mathcal{Z}\setminus\{0\}$ and continuous up to $\partial\mathcal{Z}\setminus\{z(\mu_j)\}_{j=1}^{q-1}$. Moreover, the following identities hold true:*

$$f^\sigma = \vartheta^\sigma + m_\sigma \varphi^\sigma, \qquad \sigma = \pm. \tag{3.1}$$

$$f_n^\pm(\overline{z}) = f_n^\pm(z^{-1}) = f_n^\mp(z) = \overline{f_n^\pm(z)} \quad \textit{for} \qquad |z| = 1. \tag{3.2}$$

2) $f_n^{\pm}(z)$ does not have a singularity at $z(\mu_j)$ if μ_j is not a singularity (square root singularity if μ_j coincides with the band edge) of $m_{\pm}$, otherwise, $f_n^{\pm}(z)$ can have either a simple pole at $z(\mu_j)$ if μ_j is a pole of $m_{\pm}$, or a square root singularity,

$$f_n^{\pm}(\lambda) = \pm\sigma(-1)^{q-j}\frac{iC(n)}{\sqrt{\lambda-\lambda_j^{\sigma}}} + \mathcal{O}(1), \quad \lambda \in [\lambda_{j-1}^{+}, \lambda_j^{-}], \tag{3.3}$$

if μ_j coincides with the band edge: $\mu_j = \lambda_j^{\sigma}$, $\sigma = +$ or $\sigma = -$, $j = 1, \ldots, q-1$. Here $C(n)$ is bounded and real, the factor $\sigma(-1)^{q-j}$ comes from the analytic continuation of the square root $\Omega(\lambda)$ using Definition (1.14).

The asymptotics of the function $f^{+}(z)$ are given in (4.4), (4.5).

The next lemma follows from the straightforward reformulation of the results obtained in Section 2.2 in the form stated in the definition of $\mathfrak{J}_{\nu}$.

Lemma 3.2. *If $(u, v) \in \mathfrak{X}_{\nu}$, where $\nu = 2p$ or $\nu = 2p-1$, then the Jost functions $f_0^{\pm} \in \mathfrak{J}_{\nu}$ (see Definition 2).*

3.2 Inverse scattering problem.

In this section we recall some relevant for us results from [Kh2] and [EMT]. Let $\hat{S} = \frac{\hat{f}^{-}(\lambda)}{\hat{f}^{+}(\lambda)}$. Then the scattering matrix is $S = \frac{D^{+}}{D^{-}}\hat{S}$. For each eigenvalue $\mathfrak{r}_n$ we define the norming constant m_n by

$$m_n = \sum_{j=0}^{\infty}\left(\hat{f}_j^{+}(\mathfrak{r}_n)\right)^2, \qquad n = 1, \ldots, N. \tag{3.4}$$

Introduce the scattering data for the pair of operators J, J^0 by

$$\mathcal{S}(J) = \left\{\hat{S}(\lambda),\ \text{for } \lambda \in \sigma_{\mathrm{ac}}(J^0),\ \mathfrak{r}_k,\ m_k,\ k = 1, 2, \ldots, N\right\}.$$

By the inverse scattering theory for this pair, we understand the problem of reconstructing the perturbed operator J from the scattering data and the unperturbed operator J^0.

Everywhere in this section we assumes that $(u, v) \in \mathfrak{X}_{\nu}$. We introduce the Gel'fand-Levitan-Marchenko equation for a matrix $K(n, m)$ by

$$K(n, m) + \sum_{l=n}^{+\infty} K(n, l)\mathfrak{F}_{l,m} = \frac{\delta_{nm}}{K(n, n)}, \quad m \geqslant n. \tag{3.5}$$

Here the sum in (3.5) is finite, since $(u, v) \in \mathfrak{X}_{\nu}$. The matrix $\mathfrak{F}_{l,m}$ is constructed from the scattering data $\mathcal{S}(J)$ by

$$\mathfrak{F}_{l,m} = \mathfrak{F}_{l,m}^{0} + \sum_{j=1}^{N}\frac{\hat{\psi}_l^{+}(\mathfrak{r}_j)\hat{\psi}_m^{+}(\mathfrak{r}_j)}{m_j}, \tag{3.6}$$

where

$$\mathfrak{F}^0_{l,m} = -\frac{1}{2\pi i}\int_{|z|=1} S(z)\psi_l^+(z)\psi_m^+(z)d\omega(z)$$

and

$$d\omega(z) = \prod_{j=1}^{q-1}\frac{\lambda(z)-\mu_j}{\lambda(z)-\alpha_j}\frac{dz}{z}. \tag{3.7}$$

Here $\alpha_j \in \gamma_j^+$ is the zero of $\Delta'(\lambda)$ (see Section 2.1 and (3.22) in [EMT]). Note that $\mathfrak{F}^0_{l,m} = \mathfrak{F}^0_{m,l}$ and $\mathfrak{F}^0_{l,m}$ is real. We will determine the matrix $K(n,m)$ from the Gel'fand-Levitan-Marchenko equation (3.5) and reconstruct (see (5.27) in [EMT]) a_n, b_n by

$$\frac{a_n}{a_n^0} = \frac{K(n+1,n+1)}{K(n,n)}, \quad b_n = b_n^0 + a_n^0\frac{K(n,n+1)}{K(n,n)} - a_{n-1}^0\frac{K(n-1,n)}{K(n-1,n-1)}. \tag{3.8}$$

Now we consider the Gel'fand-Levitan-Marchenko equation. From [Kh1] or [EMT], Lemma 5.1, it is known that the Jost solution f_n^+ can be represented as

$$f_n^+(z) = \sum_{m=n}^{\infty} K(n,m)\psi_m^+(z), \ |z|=1,$$

where for $(u,v)\in\mathfrak{X}_\nu$ the kernel $K(n,m)$ has finite rank and satisfies

$$K(n,m)=0, \qquad \text{for} \qquad m<n,$$

$$|K(n,m)| \leqslant C\sum_{j=[\frac{m+n}{2}]+1}^{p}\left(|u_j|+|v_j|\right), \ m>n. \tag{3.9}$$

Here the constant $C\equiv C(J^0)$ depends on the unperturbed operator J^0.

We recall the properties of the scattering data $\mathcal{S}(J)$ from [Kh2].

(I) *Function $S(\lambda)$ is continuous for $\lambda\in \operatorname{int}\partial\Gamma$, where Γ is the cut plane $\mathbb{C}\setminus\sigma_{\rm ac}(J^0)$,*

$$\overline{S(\lambda)} = S^{-1}(\lambda),\ \lambda\in\operatorname{int}\partial\Gamma,\ \textit{and}\ S(\lambda-i0)=\overline{S(\lambda+i0)},\ \lambda\in\operatorname{int}\sigma_{\rm ac}(J^0),$$

where int *stands for interior.*

(II) *The function*

$$\mathfrak{F}^0_{l,m} = -\frac{1}{2\pi i}\int_{|z|=1} S(z)\psi_l^+(z)\psi_m^+(z)d\omega(z)$$

satisfies

$$\sum_{l=0}^{\infty}\sup_{m\geqslant 0}|\mathfrak{F}^0_{l,m}| < \infty. \tag{3.10}$$

In [Kh2] this function was denoted $S(n,m)$.

(III) *Equation*

$$h_m + \sum_{k=1}^{\infty} S_{m,k} h_k = 0,\ m = 1, 2, \ldots, \tag{3.11}$$

has precisely N linearly independent solutions in $\ell^2(1, \infty)$.

(IV) *The equation $\sum_{m=-\infty}^{0} S_{l,m} g_m = g_n$ has only the zero solution in $\ell^2(-\infty, 0)$.*

(V) *The quantities a_n and b_n defined in (3.8), where $K(n, m)$ is solution to (3.5), satisfy the inequality*

$$\sum_{n=1}^{\infty} n \left(\left| \left(\frac{a_n}{a_n^0} \right)^2 - 1 \right| + |b_n - b_n^0| \right) < \infty.$$

Theorem 3.1 (Khanmamedov)**.** *If conditions (I)–(III) hold, then for every $n \in \mathbb{N}$, the Gel'fand-Levitan-Marchenko equation (3.5) has unique solution in $\ell^2(n + 1, \infty)$.*

The set $\mathcal{S}(J)$ uniquely determines J iff conditions (I)–(V) hold.

From the proof of Khanmamedov it follows that:
if $(u, v) \in \mathfrak{X}_\nu$, the bound states $\mathfrak{r}_j \in \gamma_k$, $k = 0, \ldots, q$, the norming constants m_k are given by $m_j = \sum_{n=0}^{\infty} \left(\hat{f}_n^+(\mathfrak{r}_j) \right)^2$ and S–matrix is given by $S = \frac{f_0^-(\lambda)}{f_0^+(\lambda)}$,
then conditions (I)–(V) are satisfied.

Recall that from Lemma 2.6, property (2.22), it follows that for $\mathfrak{r}_j \in \sigma_{\mathrm{bc}} \cap \gamma_k^+$ we have

$$m_j = \frac{\dot{F}(\mathfrak{r}_j)}{a_0^0 (\hat{f}_0^-(\mathfrak{r}_j))^2} \cdot (-1)^{q-k+1} 2 \sinh qh(\mathfrak{r}_j) = \frac{(\partial_\lambda \hat{f}_0^+)(\mathfrak{r}_j)}{\hat{f}_0^-(\mathfrak{r}_j)} (-1)^{q-k+1} 2 \sinh qh(\mathfrak{r}_j) > 0, \tag{3.12}$$

where $h(\mathfrak{r}_j) = \operatorname{Im} \varkappa(\mathfrak{r}_j) > 0$ (see (2.10)), as $\dot{F}(\mathfrak{r}_j) = a_0^0 (\partial_\lambda \hat{f}_0^+)(\mathfrak{r}_j) \hat{f}_0^-(\mathfrak{r}_j)$, $(-1)^{q-k} \dot{F}(\mathfrak{r}_j) = a_0^0 (-1)^{q-k} (\partial_\lambda \hat{f}_0^+)(\mathfrak{r}_j) \hat{f}_0^-(\mathfrak{r}_j) < 0$.

Now we show that the scattering data $\mathcal{S}$ can be uniquely reconstructed from any function $f \in \mathfrak{J}_\nu$ as in Definition 2 and the conditions (I)–(V) are satisfied.

The $S-$matrix and the norming constants m_j, $1 \leqslant j \leqslant N$, are then expressed in terms of the function $f \in \mathfrak{J}_\nu$ only. By abuse of notation we will keep the same letters S and m_j for the functions expressed in f.

Using Theorem 3.1 this will imply that the function $f \in \mathfrak{J}_\nu$ uniquely determines J.

Lemma 3.3. *Let $f = P_1 + m_+ P_2 \in \mathfrak{J}_\nu$, $f_- = P_1 + m_- P_2$, $P(\lambda) = \varphi_q f f_-$ and $\sigma_{\mathrm{bs}}(f) = \{\mathfrak{r}_j\}_{j=1}^{N} \in \Lambda_1$ be as in Definition 2. We define m_j, $j = 1, \ldots, N$, by*

$$m_j = \frac{\dot{P}(\mathfrak{r}_j)}{a_0^0 (\hat{f}_-(\mathfrak{r}_j))^2} \cdot (-1)^{q-k+1} 2 \sinh qh(\mathfrak{r}_j), \qquad \mathfrak{r}_j \in \gamma_k^+, \tag{3.13}$$

where $\hat{f}_- = D^{\mathrm{e}} D^- f_-$, and $S(\lambda) := \frac{f_-(\lambda)}{f(\lambda)}$. Then conditions (I)-(V) are satisfied.

Proof. (I) Recall that by (1.20) $S(\lambda) = \frac{\overline{f_0^+(\lambda)}}{f_0^+(\lambda)} = \frac{f_0^-(\lambda)}{f_0^+(\lambda)}$, and then it follows

$$\overline{S(\lambda)} = S^{-1}(\lambda),\ \lambda \in \operatorname{int} \partial\Gamma,\ \text{and } S(\lambda - i0) = \overline{S(\lambda + i0)},\ \lambda \in \operatorname{int} \sigma_{\rm ac}(J^0),$$

(II) In the next section we prove that if $\{\lambda_j\}_{j=1}^{\kappa} \in \sigma_{\rm st}(f)$ then the sum (3.10) is finite and the condition is trivially satisfied.

(III) Khanmamedov [Kh2] showed that the number of linearly independent solutions in $\ell^2(1,\infty)$ of (3.11) coincides with that of linearly independent functions of the form $\frac{C_k \hat{f}_0^+(\lambda)}{\partial_\lambda \hat{f}_0^+(\mathfrak{r}_j)(\lambda - \mathfrak{r}_j)}$. For $\{\lambda_j\}_{j=1}^{\kappa} \in \sigma_{\rm st}(f)$ as in Introduction it follows that the values $\mathfrak{r}_j \in \mathbb{R}\backslash\sigma_{\rm ac}(J^0)$, $1 \leqslant j \leqslant N$, are distinct and the norming constants m_j, $1 \leqslant j \leqslant N$, are positive, which implies that the number of linearly independent functions is precisely N.

(IV) The condition is proved similarly to (III).

(V) For $(u,v) \in \mathfrak{X}_\nu$ and a_n, b_n defined in (1.7) or for $\{\lambda_j\}_{j=1}^{\kappa} \in \sigma_{\rm st}(f)$ for $f \in \mathfrak{J}_\nu$, as in Definition 2, this sum is finite as shown in the next section. ∎

3.3 Inverse resonance problem.

We prove here the Theorems 1.3-1.5.

Proof of Theorem 1.3.

We will prove the following: *The mapping $\mathscr{F} : \mathfrak{X}_\nu \to \mathfrak{J}_\nu$ given by*

$$(u,v) \to f_0^+(u,v) \in \mathfrak{J}_\nu,$$

is one-to-one and onto. Recall that $\nu \in \{2p-1, 2p\}$. In particular, a pair of coefficients in $\mathfrak{X}_\nu$ is uniquely determined by its bound states and resonances.

Uniqueness. In the first part of this paper we proved that to any $(u,v) \in \mathfrak{X}_\nu$ we can associate the Jost function $f \in \mathfrak{J}_\nu$. Let $\sigma_{\rm st}(f)$ be the class of points on Λ specified in Definition 2, $f_- = P_1 + m_- P_2$, the bound states $\mathfrak{r}_j \in \sigma_{\rm bs}(f) \subset \Lambda_1$, the norming constants m_j by (3.13), $j = 1, \ldots, N$, and the scattering matrix $S = \frac{f_-}{f}$. Then conditions (I)–(V) of Theorem 3.1 are satisfied and these data determine $(u,v) \in \mathfrak{X}_\nu$ uniquely. Then we have that the mapping $(u,v) \to f_0^+(u,v) \in \mathfrak{J}_\nu$ is an injection.

Surjection. We will show that the mapping $(u,v) \to f_0^+(u,v) \in \mathfrak{J}_\nu$ is surjective. Let $f \in \mathfrak{J}_\nu$ as in Definition 2.

Then we define m_j, $j = 1, \ldots, N$, by (3.13) and $\hat{S} = \frac{\hat{f}_-}{\hat{f}}$, where $\hat{f} = D^+ D^{\rm e} f$, $\hat{f}_- = D^- D^{\rm e} f_-$. Lemma 3.3 shows that the set of quantities $\mathcal{S} = \{\hat{S}(\lambda),\ \text{for } \lambda \in \sigma_{\rm ac}(f),\ z_k,\ m_k,\ k = 1, 2, \ldots, N\}$ is unique scattering data verifying conditions (I)–(V). Then by solving the Gel'fand-Levitan-Marchenko equation and applying Theorem 3.1 we get the unique coefficients (u,v). We need to show that $(u,v) \in \mathfrak{X}_\nu$.

We have

$$\mathfrak{F}^0_{l,m} = -\frac{1}{2\pi i}\int_{|z|=1} S(z)\psi_l^+(z)\psi_m^+(z)d\omega(z),$$
$$= -\frac{1}{2\pi i}\int_{\partial\Gamma} \hat{S}(\lambda)\hat{\psi}_l^+(\lambda)\hat{\psi}_m^+(\lambda)\frac{d\lambda}{2(\Delta^2(\lambda)-1)^{1/2}},$$

Observe that $d\omega$ is meromorphic on $\mathcal{Z}_1$ with simple pole at $z=0$. In particular, there are no poles at $z(\alpha_j)$. To evaluate the integral we use the residue theorem. Take a closed contour in $\mathcal{Z}_1$ and let this contour approach $\partial\mathcal{Z}_1$. The function $S(z)\psi_l^\pm(z)\psi_m^\pm(z)$ is continuous on $\{|z|=1\}\setminus\{z(E_j)\}$ and meromorphic on $\mathcal{Z}_1$ with simple poles at $z(\mathfrak{r}_j)$ and eventually a pole at $z=0$.

We have

$$S(z) = z^{-\nu}\left(1+\mathcal{O}(z)\right), \qquad \psi_l^+\psi_m^+ = z^{l+m}\left(1+\mathcal{O}(z)\right), \qquad \text{as} \quad z\to 0.$$

Suppose $l+m\geqslant \nu+1$ ($+1$ is due to singularity of z^{-1} in $d\omega$). Then the integrand is bounded near $z=0$ and we apply the residue theorem to the only poles at the eigenvalues.

We have ([EMT], (3.23))

$$\frac{dz}{d\lambda} = z\frac{\prod_{j=1}^{q-1}(\lambda-\alpha_j)}{2(\Delta^2(\lambda)-1)^{1/2}}$$

and, if $z_j = z(\mathfrak{r}_j)$, then $\mathrm{Res}_{z=z_j}F(z) = z'(\mathfrak{r}_j)\mathrm{Res}_{\lambda=\mathfrak{r}_j}F(z(\lambda))$.

We get

$$\mathfrak{F}^0_{l,m} = -\sum_{j=1}^{N}\mathrm{Res}_{\mathfrak{r}_j}\left(\frac{\hat{S}(\lambda)\hat{\psi}_l^+(\lambda)\hat{\psi}_m^+(\lambda)}{2(\Delta^2(\lambda)-1)^{1/2}}\right),$$

where $(\Delta^2(\lambda)-1)^{1/2} = i\Omega(\lambda)$. Now

$$\hat{S}(\lambda) = \frac{\hat{f}_-(\mathfrak{r}_j)}{\partial_\lambda\hat{f}(\mathfrak{r}_j)(\lambda-\mathfrak{r}_j)}\left(1+\mathcal{O}(\lambda-\mathfrak{r}_j)\right) \qquad \text{as} \quad \lambda\to\mathfrak{r}_j.$$

Then, using that $2(\Delta^2(\lambda)-1)^{1/2} = (-1)^{q-k+1}2\sinh qh(\lambda)$ (see (2.10)) and (3.13), we get

$$\mathfrak{F}^0_{l,m} = -\sum_{j=1}^{N}\frac{\hat{f}_-(\mathfrak{r}_j)}{\partial_\lambda\hat{f}(\mathfrak{r}_j)2(\Delta^2(\mathfrak{r}_j)-1)^{1/2}}\hat{\psi}_l^+(\mathfrak{r}_j)\hat{\psi}_m^+(\mathfrak{r}_j) = -\sum_{j=1}^{N}m_j^{-1}\hat{\psi}_l^+(\mathfrak{r}_j)\hat{\psi}_m^+(\mathfrak{r}_j)$$

Then equation (3.6) implies

$$\mathfrak{F}_{l,m} = \mathfrak{F}^0_{l,m} + \sum_{j=1}^{N}m_j^{-1}\hat{\psi}_l^+(\mathfrak{r}_j)\hat{\psi}_m^+(\mathfrak{r}_j) = 0, \quad l+m\geqslant\nu+1,$$

and the Gel'fand-Levitan-Marchenko equation

$$K(n,m) + \sum_{l=n}^{+\infty}K(n,l)\mathfrak{F}_{l,m} = \frac{\delta_{nm}}{K(n,n)}, \quad m\geqslant n,$$

implies that the kernel of the transformation operator $K(n,m)$, satisfies

$$K(n,m) = \frac{\delta_{nm}}{K(n,n)}, \quad m \geqslant n, \quad m+n \geqslant \nu+1.$$

Thus we get

$$\text{If } 2n \geqslant \nu+1, \text{ then } K(n,n) = \pm 1; \quad \text{if } n+m \geqslant \nu+1, \ m \neq n, \text{ then } K(n,m) = 0.$$

We recall (3.8)

$$\frac{a_n}{a_n^0} = \frac{K(n+1,n+1)}{K(n,n)}, \quad v_n = a_n^0 \frac{K(n,n+1)}{K(n,n)} - a_{n-1}^0 \frac{K(n-1,n)}{K(n-1,n-1)}.$$

Then, as $a_n > 0$, $a_n^0 > 0$, we get $a_n = a_n^0$ for $n \geqslant p+1$, if $\nu = 2p$ (or for $n \geqslant p$ if $\nu = 2p-1$). Moreover, we get $v_n = 0$ for $2n-1 \geqslant 2p+1$ (or $2n-1 \geqslant 2p$) which both implies $n \geqslant p+1$ and $v_p \neq 0$, if $\nu = 2p-1$. This yields surjection.

From (3.9) we get also that if $(u,v) \in \mathfrak{X}_\nu$ then $K(n,m) = 0$ for $n+m \geqslant 2p$. ■

Proof of Theorems 1.4 and 1.5. Recall that for any $\lambda \in \Lambda$ the map $\lambda \mapsto \widetilde{\lambda} \in \Lambda_1$ denotes the projection to the first sheet and Λ_1 is identified with $\Gamma = \mathbb{C} \setminus \sigma_{\rm ac}(J^0)$. Note that from Lemma 2.8 it follows that, due to the assumption (1.30), the (projection of) set of solutions of the equation $S = 1$ is the set of all zeros of polynomial φ_0^+. Recall that the polynomials F, φ_0^+ have orders $\kappa = \nu + q - 1$ and $\nu - 1$, respectively. We denote their sets of zeros by $\mathrm{Zeros}\,(F) = \{\lambda_j\}$ and $\mathrm{Zeros}\,(\varphi_0^+) = \{\omega_j\}$ respectively. Now for given $\mathrm{Zeros}\,(F)$, $\mathrm{Zeros}\,(\varphi_0^+)$ and the constants c_1, c_2, we can reconstruct the unique polynomials $F(\lambda) = C_1 \prod_{j=1}^{\nu+q-1}(\lambda - \lambda_j)$, $\varphi_0^+(\lambda) = C_2 \prod_{j=1}^{\nu-1}(\lambda - \omega_j)$. We need to distinguish between projections to the complex plane of the bound states and the resonances.

Let $\sigma_1 = \{\lambda_j\}_{j=1}^{N_1}$, $N_1 \leqslant N$, be the set of zeros of F such that:
1) $\sigma_1 \cap \widetilde{\sigma}_{\rm bs}(J_0) = \emptyset$;
2) $\sigma_1 \in \bigcup_0^q \gamma_j$ and if $\lambda_0 \in \widetilde{\sigma}_1 \bigcap \gamma_n$ for some $n = 0, \ldots q$, then $(-1)^{q-n}\dot{F}(\lambda_0) < 0$.

Let $\sigma_2 = \{\lambda_j\}_{j=N_1+1}^{\kappa_1}$, $\kappa_1 \leqslant \kappa$, be the set of zeros of F such that:
1) $\sigma_2 \cap (\widetilde{\sigma}_{\rm r}(J_0) \cup \widetilde{\sigma}_{\rm vs}(J_0)) = \emptyset$;
2) if $\lambda_j \in \sigma_2$ is real, then $\lambda_j \in \gamma_n$, for some $n = 0, \ldots q$, and $(-1)^{q-n}\dot{F}(\lambda_j) \geqslant 0$.

We consider the following polynomial interpolation problem:

$$\begin{aligned} \vartheta_0^+(\lambda_j) &= -m_+(\lambda_j)\varphi_0^+(\lambda_j) \quad \text{for} \quad \lambda_j \in \sigma_1, \ j = 1, \ldots, N_1, \\ \vartheta_0^+(\lambda_j) &= -m_-(\lambda_j)\varphi_0^+(\lambda_j) \quad \text{for} \quad \lambda_j \in \sigma_2, \ j = N_1+1, \ldots, \kappa_1. \end{aligned} \tag{3.14}$$

Suppose that each zero $\lambda_j \in \sigma_1 \cup \sigma_2 \subset \mathrm{Zeros}\,(F)$, $j = 1, \ldots, \kappa_1$, is simple. Then we have $\nu \leqslant \kappa_1 \leqslant \nu + q - 1$ and it is well known (see for example the book of Kendell A. Atkinson [A]) that the polynomial interpolations problem (3.14) defines unique polynomial ϑ_0^+ of order $\nu - 2$, and therefore the unique Jost function $f_0^+ = \vartheta_0^+ + m_+\varphi_0^+$. ■

4 Asymptotics of the Jost function on the unphysical sheet.

In this section we obtain asymptotics of the Jost solutions $f^{\pm}$ and prove Lemma 2.3. The asymptotics of $f^{+}(\lambda)$ as $\lambda \in \Lambda_1$ and $\lambda \to \infty$ are well known (see for example [T]). We obtain the asymptotics of $f^{+}_{p-n}(\lambda)$ as $\lambda \in \Lambda_2$ and $\lambda \to \infty$, which is equivalent to the asymptotics of f^{-}_{p-n} for $\lambda \in \Lambda_1$. In this section we will not assume $A = 1$. We will omit the upper indexes $^{\pm}$ as much as possible. We make use of (2.12):

$$f_{p+1} = \psi_{p+1}, \quad f_p = \frac{a_p^0}{a_p}\psi_p.$$

Put $\Phi(j) = \frac{\psi_{j+1}}{\psi_j}$. Now (see [T]) we have

$$\psi_p = \prod_{j=0}^{p-1}{}^{*}\Phi(j) = \begin{cases} \prod_{j=0}^{p-1}\Phi(j) & \text{for } p > 0 \\ 1 & \text{for } p = 0 \\ \prod_{j=0}^{p-1}(\Phi(j))^{-1}. & \text{for } p < 0, \end{cases}$$

If $\psi = \psi^{\pm}$ then $\Phi(0) = \Phi^{\pm}(0) = m_{\pm}$ and we have (see [T])

$$\Phi^{\pm}(\lambda, n) = \left(\frac{a^0(n)}{\lambda}\right)^{\pm 1}\left(1 \pm \frac{b^0(n + {}^{1}_{0})}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right), \qquad \lambda \to \infty,$$

where $a_n^0 \equiv a^0(n)$, $b_n^0 \equiv b^0(n)$. Put $\Psi(n) = \Phi^{-1}(n)$, then

$$\Psi^{\pm}(\lambda, n) = \left(\frac{a^0(n)}{\lambda}\right)^{\mp 1}\left(1 \mp \frac{b^0(n + {}^{1}_{0})}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right), \qquad \lambda \to \infty.$$

By iterating the Jacobi equation (2.11) we get

$$\begin{aligned}
f_{p-1} &= \frac{(\lambda - b_p)a_p^0\psi_p - a_p^2\psi_{p+1}}{a_p a_{p-1}} = \frac{\psi_{p+1}}{a_p a_{p-1}}\left((\lambda - b_p)a_p^0\Psi(p) - a_p^2\right); \\
f_{p-2} &= \frac{(\lambda - b_{p-1})a_{p-1}f_{p-1} - a_{p-1}^2\frac{a_p^0}{a_p}\psi_p}{a_{p-1}a_{p-2}} = \\
&= \frac{\psi_{p+1}}{a_p a_{p-1} a_{p-2}}\left((\lambda - b_{p-1})\left[(\lambda - b_p)a_p^0\Psi(p) - a_p^2\right] - a_{p-1}^2 a_p^0\Psi(p)\right); \\
f_{p-3} &= \frac{(\lambda - b_{p-2})a_{p-2}f_{p-2} - a_{p-2}^2\frac{\psi_{p+1}}{a_p a_{p-1}}\left((\lambda - b_p)a_p^0\Psi(p) - a_p^2\right)}{a_{p-2}a_{p-3}} = \\
&= \frac{\psi_{p+1}}{a_p \dots a_{p-3}}\left((\lambda - b_{p-2})\left[(\lambda - b_{p-1})\left[(\lambda - b_p)a_p^0\Psi(p) - a_p^2\right] - a_{p-1}^2 a_p^0\Psi(p)\right] -\right. \\
&\quad \left. - a_{p-2}^2\left((\lambda - b_p)a_p^0\Psi(p) - a_p^2\right)\right).
\end{aligned}$$

Now we use that $\Psi(p) \equiv \Psi^-(\lambda, p) = \frac{a_p^0}{\lambda}\left(1 + \frac{b_p^0}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right)$ as $\lambda \to \infty$. Then we get

$$\psi_{p+1} \equiv \psi_{p+1}^-(\lambda) = \frac{\lambda^{p+1}}{A_p}\left(1 - \frac{1}{\lambda}\sum_{j=0}^{p} b_j^0 + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right), \qquad \lambda \to \infty,$$

where $A_p = \prod_{j=0}^{p} a_j^0$. We have

$$(\lambda - b_p)a_p^0\Psi(p) - a_p^2 = ((a_p^0)^2 - a_p^2) + \frac{(a_p^0)^2}{\lambda}(b_p^0 - b_p) + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

and get

$$\begin{aligned} f_{p-n} &= \frac{\lambda^{p+n}}{A_p\prod_{j=p-n}^{p} a_j} \\ &\cdot\left(((a_p^0)^2 - a_p^2) + \frac{1}{\lambda}\left[-((a_p^0)^2 - a_p^2)(\sum_{j=0}^{p} b_j^0 + \sum_{j=p-n+1}^{p-1} b_j) - (a_p^0)^2 v_p\right] + \frac{\mathcal{O}(1)}{\lambda^2}\right), \\ f_0(\lambda) &= \frac{c_1\lambda^{2p}}{A_p} \\ &\cdot\left(((a_p^0)^2 - a_p^2) + \lambda^{-1}\left[-((a_p^0)^2 - a_p^2)(\sum_{j=0}^{p} b_j^0 + \sum_{j=1}^{p-1} b_j) - (a_p^0)^2 v_p\right] + \frac{\mathcal{O}(1)}{\lambda^2}\right). \end{aligned} \tag{4.1}$$

If $a_p = a_p^0$, then

$$f_0(\lambda) = -\frac{c_1(a_0^p)^2 v_p}{A_p}\lambda^{2p-1} + \mathcal{O}\left(\lambda^{2p-2}\right).$$

Multiplying

$$\begin{aligned} \varphi_q &= \frac{\lambda^{q-1}}{A_{q-1}} + \mathcal{O}(\lambda^{q-2}), \\ f_n^+ &= \alpha_n^+\frac{\prod_{j=0}^{n-1}{}^*a_j}{\lambda^n}\left[1 + \frac{1}{\lambda}\left(-\sum_{j=1}^{p} v_j + \sum_{j=1}^{n}{}^*b_j\right) + \frac{\mathcal{O}(1)}{\lambda^2}\right], \\ (f_n^+)^* &= \frac{\lambda^{2p-n}}{\prod_{j=n}^{p} a_j A_p} \\ &\cdot\left(((a_p^0)^2 - a_p^2) + \lambda^{-1}\left[(a_p^2 - (a_p^0)^2)(\sum_{j=0}^{p} b_j^0 + \sum_{j=n+1}^{p-1} b_j) - (a_p^0)^2 v_p\right] + \frac{\mathcal{O}(1)}{\lambda^2}\right), \end{aligned}$$

and using $\alpha_n^+ = \prod_{j=n}^{p}\frac{a_j^0}{a_j}$, we get

$$F_n(\lambda) = \varphi_q f_n^+(f_n^+)^* = \frac{c_1^2\lambda^{2(p-n)+q-1}}{A_{q-1}}\left(((a_p^0)^2 - a_p^2) + \mathcal{O}(\lambda^{-1})\right), \quad \text{if} \quad u_p \neq 0, \tag{4.2}$$

$$F_n(\lambda) = \varphi_q f_n^+ (f_n^+)^* = \frac{c_1^2 \lambda^{2(p-n)+q-2}}{A_{q-1}} \left(-(a_p^0)^2 v_p + \mathcal{O}(\lambda^{-1}) \right), \quad \text{if} \quad u_p = 0, \quad v_p \neq 0, \tag{4.3}$$

where $c_1(n) = (\prod_{j=n}^{p} a_j)^{-1}$. On the Riemann surface $\mathcal{Z}$ as in Section 3.1 we get

$$f_0^+ = \alpha_0^+ + \mathcal{O}(z), \text{ as } z \to 0, \tag{4.4}$$

$$\begin{aligned} f_0^+ = & \frac{c_1 A^{\frac{2p}{q}} z^{2p}}{A_p} \\ & \cdot \left(((a_p^0)^2 - a_p^2) + \frac{A^{-\frac{1}{q}}}{z} \left[-((a_p^0)^2 - a_p^2)(\sum_{j=0}^{p} b_j^0 + \sum_{j=1}^{p-1} b_j) - (a_p^0)^2 v_p \right] + \frac{\mathcal{O}(1)}{z^2} \right), \\ & \text{as } z \to \infty. \end{aligned} \tag{4.5}$$

References

[A] Atkinson, K.A. *An Introduction to Numerical Analysis* 2nd ed., John Wiley and Sons, 1989.

[BGGK] Bättig, D.; Grebert, B.; Guillot, J.-C.; Kappeler, T. *Fibration of the phase space of the periodic Toda lattice*. J. Math. Pures Appl., 72 (1993), no. 6, 553–565.

[BE] A Boutet de Monvel, I. Egorova. *Transformation operator for jacobi matrices with asymptotically periodic coefficients*. J. of Difference Eqs. Appl., 10 (2004), 711–727.

[BKW] Brown, B.; Knowles, I.; Weikard, R. *On the inverse resonance problem*. J. London Math. Soc. (2) 68 (2003), no. 2, 383–401.

[DS1] D. Damanik, B. Simon. *Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegö asymptotics*. Invent. Math., 165(1) (2006), 1–50.

[DS2] D. Damanik, B. Simon. *Jost functions and Jost solutions for Jacobi matrices, II. Decay and analyticity*. Int. Math. Res. Not., Art. ID 19396, (2006).

[EMT] I. Egorova, J. Michor, G. Teschl. *Scattering Theory for Jacobi operators with quasi-periodic background*. Commun. Math: Phys., 264 (2006), 811–842.

[F1] N. Firsova. *Resonances of the perturbed Hill operator with exponentially decreasing extrinsic potential*. Mat. Zametki, 36 (1984), 711–724.

[Fr] R. Froese. *Asymptotic distribution of resonances in one dimension*. J. Diff. Eq., 137 (1997), 251–272.

[IK1] A. Iantchenko, E. Korotyaev. *Schrödinger operator on the zigzag half-nanotube in magnetic field*. Math. Model. Nat. Phenom., Spectral Problems, 5(4) (2010), 175–197.

[IK2] A. Iantchenko, E. Korotyaev. *Periodic Jacobi operators with finitely supported perturbations*. arXiv.

[IK3] A. Iantchenko, E. Korotyaev. *Inverse resonance problem for periodic Jacobi operators with finitely supported perturbations on the line*. arXiv.

[Kh1] Ag. Kh. Khanmamedov. *The inverse scattering problem for a perturbed difference Hill equation*. Mathematical Notes, 85(3) (2009), 456–469.

[Kh2] Ag. Kh. Khanmamedov. *The inverse scattering problem for a Schrödinger difference operator with asymptotically periodic coefficients defined on the half-axis. (Russian)*. Dokl. Akad. Nauk, 409(4) (2006), 451–454.

[KM] F. Klopp, M. Marx. *The width of resonances for slowly varying perturbations of one-dimensional periodic Schrödinger operators*. Seminaire: EDP. 2005-2006, Exp. No. IV, Ecole Polytech., Palaiseau, (2006).

[K1] E. Korotyaev. *Resonance theory for perturbed Hill operator*, will be published in Asympt. Anal.

[K2] E. Korotyaev. *Inverse resonance scattering for Jacobi operators*. arXiv.

[K3] Korotyaev, E. *Gap-length mapping for periodic Jacobi matrices*, Russ. J. Math. Phys. 13(2006), no.1, 64-69.

[K4] E. Korotyaev. *Inverse resonance scattering on the real line*. Inverse Problems, 21(1) (2005), 325–341.

[K5] E. Korotyaev. *Inverse resonance scattering on the half-line*. Asymptotic Analysis, 37(3/4) (2004), 215–226.

[K6] Korotyaev, E. *Stability for inverse resonance problem*, Int. Math. Res. Not. 2004, no. 73, 3927–3936.

[KKu] Korotyaev, E.; Kutsenko, A. *Marchenko-Ostrovski mappings for periodic Jacobi matrices*. Russ. J. Math. Phys. 14(2007), no 4, 448-452.

[KS] E. Korotyaev; K., M., Schmidt, *On the resonances and eigenvalues for a 1D half-crystal with localised impurity*, will be published in J. Reine Angew. Math..

[Mo] Pierre van. Moerbeke. *The Spectrum of Jacobi Matrices*. Inventiones Math., 37 (1976), 45–81.

[P] L. Percolab. *The inverse problem for the periodic Jacobi matrix*. Teor. Funk. An. Pril., 42 (1984), 107–121.

[S] B. Simon. *Resonances in one dimension and Fredholm determinants*. J. Funct. Anal., 178(2) (2000), 396–420.

[T] G. Teschl. *Jacobi operators and completely integrable nonlinear lattices*. Providence, RI: AMS, (2000) (Math. Surveys Monographs, V. 72.)

[Z] M. Zworski. *Distribution of poles for scattering on the real line*. J. Funct. Anal., 73 (1987), 277–296.

[Z1] M. Zworski. *A remark on isopolar potentials*. SIAM, J. Math. Analysis, 82(6) (2002), 1823–1826.